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A PROPOSAL FOR A FRACTAL -BASED 'DYNAMIC' PROGRAM: THE PYTHAGOREAN TREE STRUCTURE GENERATED THROUGH 'INSTRUMENTAL' SCHEMATA

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ABSTRACT

This document primarily investigates the structure of the Pythagorean Tree, integrating my previous research and writings with earlier insights and contemporary ideas related to its fractal nature, which is produced through instrumental schemata. To illustrate the concept of instrumental schemata, I will present the example of Generator tools that can be created within dynamic geometry software environments. I will briefly discuss the van Hiele theory, which is crucial for classifying students based on their understanding and interpretation of geometric figures. In one section, I will investigate a fresh perspective on the Pythagorean tree, concentrating on the figures that arise when one carefully observes the empty spaces formed between the branches of the tree. Finally, I will delineate the essential elements of an innovative Fractal-based Dynamic Program (FDP) that I designed, developed and implemented. I will propose its effective implementation, as it has the potential to serve as an informal curriculum centered on the principles of transformation geometry and fractals for educational projects. It is anticipated that educators will regard the incorporation of fractal geometry into the standard curriculum as a beneficial didactic and pedagogic framework for fostering students' curiosity and demonstrating the dynamic character of the field.

Keywords: Fractal, Pythagorean Tree, Dynamic Geometry Software, van Hiele Geometric Thinking Levels, Transformation Geometry, Instrumental social schema

1.0 INTRODUCTION: THE PYTHAGOREAN TREE FRACTAL STRUCTURE

This study will begin by highlighting my significant engagement with the concept of fractals, which started in 2005, while I was preparing my Master's thesis as part of the Postgraduate Program in Didactic and Methodology of Mathematics. In my dissertation research, I conducted a thorough investigation of the construction of fractal structures, including the Sierpinski triangle, the Pythagorean tree, and various spirals such as the Golden spiral, the Archimedean spiral, and the Baravelle spiral. My main objective was to establish an instrumental, conceptual, and experimental connection between these fractal constructions and the students' perception and comprehension of mathematical concepts such as sequences, geometric progressions, limits, and infinitesimals. Additionally, I aimed to explore how students across different levels of education—elementary, middle, and high school—could conceptually understand these ideas through the use and interaction with a DGS software, The Geometer's Sketchpad (Jackiw, 1991). In the process of constructing fractals, I developed and utilized customized tools within dynamic geometry software, with the objective of creating a

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layered structure—an endeavor to illustrate the concept of self-similarity in my lesson studies. This process led me to conceptualize the notion of ‘schema’ (e.g., Patsiomitou, 2008c) in the sense articulated by Vergnaud (1998): a dynamic concept that continuously adapts to include new components—a cognitive structure that is, in essence, 'self-similar'. Moreover, the iterative process provided by the software environment enabled me to conceive the notion of *linking visual active representations* (e.g., Patsiomitou, 2008a, b, 2012 a, b), whereas the creation of fractal objects could be directly linked to the visual ‘alive’ tabulation of measurements and calculations. This, in turn, provided a natural pathway to concepts such as sequences, geometric progressions, limits, and infinitesimals, through multiple dynamic and active representations (e.g., see for example my research study on fractal objects at ICTMT8 (Patsiomitou, 2007a) based on my Master’s thesis published 2005). This paper mainly investigates the structure of the Pythagorean Tree, incorporating my previous research findings and writings, along with recent perspectives concerning its fractal structure, which is produced through ‘*instrumental schemata'* (Patsiomitou, 2025).

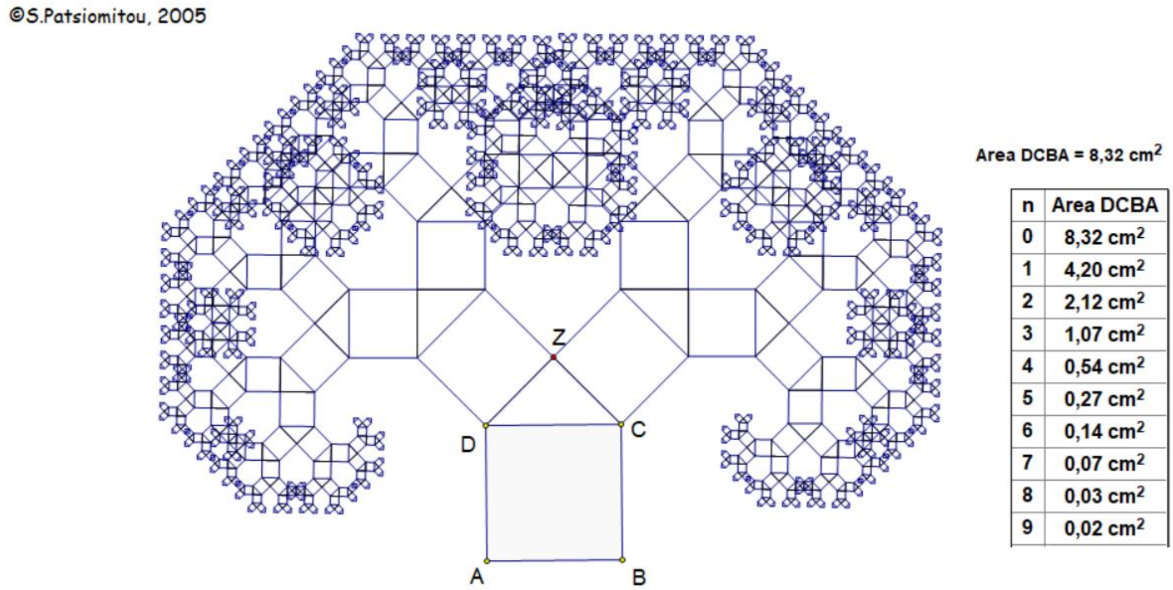

Figure 1. A Pythagorean tree figure linked with a sequence of sequential squares’ areas (e.g., Patsiomitou, 2005a, b, 2007a, b, 2009c, 2022a)

The implementation of 'dynamic' fractal structures in the classroom as part of an FDP (Fractal-based Dynamic Program) proved to be highly engaging for students. They produced stunning constructions as part of the FDP that conceptualized and directed, titled 'Omilos for Fractals' (also known as *Fractals Group: From Zero to Infinity*). The development and enhancement of the FDP content was informed by my experiences with students in a project-based learning environment throughout the academic years 2011-12, 2012-13, and 2013-14. What became increasingly important to me was the integration of fractals into the school curriculum. I recognized the experimental potential of fractals to meaningfully engage with concepts from Algebra, Geometry, and Calculus, through the construction of fractals and the study of their algebraic, geometric, and calculus-related properties. I was able to rediscover/reinvent for my students many important ideas of the existing mathematics curriculum across several grade levels. Section 10 of this document introduces the FDP, that has the potential to serve as an informal curriculum aimed at engaging students with mathematical concepts beyond standard school hours or in addition to the official mathematics curriculum. The instructional resources utilized in the FDP incorporate several chapters from my book, *Learning Mathematics with Geometer's Sketchpad v4* (Patsiomitou, 2009b, c). The book has been updated in 2022 under the title *Conceptual and instrumental trajectories using linking visual active representations created with the Geometer's Sketchpad* (Patsiomitou, 2022a). This book integrates and elaborates on my earlier research, which I have presented at conferences in Greece and published in academic journals (e.g., Patsiomitou, 2005a, b, 2006a, b, c, d, e, f, g, 2007a, b, c, d, e 2008a, b, c, d, e, f, g, h, 2009a, d, e, f, g, h). For the FDP, I developed instructional plans, activity sheets, and worksheets related to *Transformation Geometry*, intended for use in an interactive learning environment. The activities were meticulously tailored to align with the cognitive levels, developmental stages, and ages of the students, employing the van Hiele model (e.g., Fuys et al., 1984). As a result of my research, I proposed the development of a 'Dynamic' curriculum framework grounded in the Fractal-based Dynamic Program (FDP). I have briefly presented and published (Patsiomitou, 2016 a, b, c in Greek) the sequential phases of the Fractal-based Dynamic Program (FDP) that I implemented in the classroom to provide valuable insights for other mathematics educators.

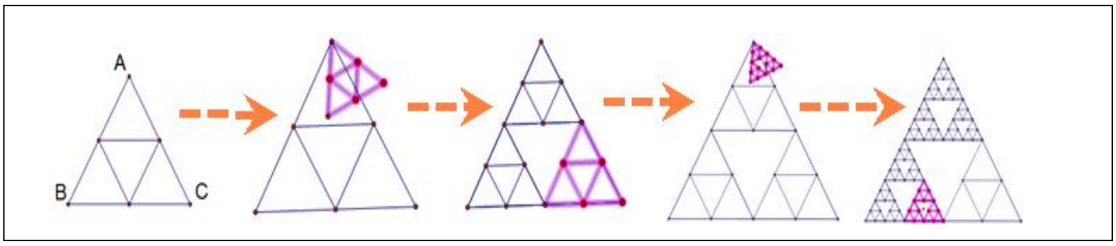

Figure 2. Sierpinski's linking visual active representations via the utilization of sequential custom tools (Patsiomitou, 2005 a, b, 2014, p. 26**)**

This paper, as I mentioned above, investigates the Pythagorean Tree structure. The Pythagorean Theorem, as presented in most school textbooks, is typically illustrated through a visual proof involving a right-angled triangle. This triangle has two squares on its perpendicular sides, resembling branches, and a third square on the hypotenuse, symbolizing the trunk of a 'tree'. If this visual configuration is used to iteratively reconstruct the visual proof, it results in a fractal structure, wherein the shape's pattern remains consistent across successive, scaled-down iterations of the construction. Moreover, during this sequential process, the shape appears to rotate in order to generate the branches of the original "tree." Thus, replications of the initial structure emerge—branching visual demonstrations of the Pythagorean Theorem—that progressively is a repetition of the process itself. Consequently, both structurally and instrumentally, the dimensions of the figures and subfigures continuously decrease. This leads to the following question: What is the limiting value of the area of the square/or other figures located at the end of each branch? The research questions I investigated throughout the study pertain to the notions of geometric sequences, limits, infinitesimals, the nth partial sum of a geometric progression, as well the notion of similarity and self-similarity. Shriki and Nutov (2016) examine the idea of self-similarity across an infinite series of iterations in their research. They characterize fractals as described in the subsequent excerpt: "*A Fractal is a geometrical object that displays self-similarity i.e. a recurring pattern or structure that manifests at* *progressively smaller scales. Fractals are constructed through a recursive process where each phase of its construction is a repetition of the process itself. This recursive process consists of an infinite number of iterations termed 'phases', where 'phase 0' designates the original shape from which the process started*" (Shriki &Nutov, 2016, p. 38). Löfstedt (2008) in his Master's thesis *Fractal Geometry, Graph and Tree constructions* discusses the origins of fractals in the following excerpt (p.21): *Benoit Mandelbrot (1975) coined the term Fractal, and described it as follows: A [fractal is a] rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. The word is derived from the Latin word fractus meaning broken, and is a collective name for a diverse class of geometrical objects, or sets, holding most of, or all of the following properties (Falconer, 1990).*

The Pythagorean tree's structure, originating from the unique generating node (3,4,5), was credited to B. Berggren in 1934 and J.M. Barning in 1963. Based on the Pythagoras theorem equation the Pythagorean Tree was initially presented by Dutch mathematics educator Albert E. Bosman in 1942. (Refer to https://en.wikipedia.org/wiki/Pythagoras_tree_(fractal) and the research by Ali Taghavi (2024) at https://arxiv.org/html/2403.17966v1.) Luis Teia (2016) discusses the Pythagorean tree structure, stating that:

> "*The Pythagoras' tree presented by Berggren in 1934 has stood still and strong for almost a century, but probably it is even older. […] Ultimately, when one looks at the Pythagoras' tree, one looks at a 'tree'". (p.38)*

This paper discusses the case of the Pythagorean tree with an isosceles right-angled triangle that is attached to the side of the original square in the Pythagorean tree. The Pythagorean tree is a self-similar fractal structure with ever-decreasing geometric dimensions, but maintains its shape at every scale. I will analyze the structure of the Pythagorean tree, which is based on a right-angled triangle with sides 3, 4, and 5, in a forthcoming publication.

In the following section, I will briefly address the Van Hiele theory, which is significant for categorizing students based on their perception and comprehension of geometric figures. Additionally, I will discuss the concept of 'schema' as articulated in Vergnaud's studies, in relation to the idea of the *instrumental schema*, which I have recently introduced in my latest publication (Patsiomitou, 2025). To exemplify this concept, I will provide the instance of generator tools that can be developed within dynamic geometry software environments. In one section, I will examine a novel viewpoint on investigating the Pythagorean tree, focusing on the shapes that emerge when one meticulously observes with great attention to detail the spaces created between the tree's branches. I concur with Kinach (2014) in stating that "*this is a matter of attention. Attention shifts from interacting with the actual problem while problem solving to interacting with successive representations of it*" (p. 434). Lastly, I will outline the key components of the innovative Fractal-based Dynamic Program [FDP], and provide guidance for its effective implementation, as it holds the promise of functioning as an informal curriculum focused on the principles of transformation geometry and fractals for after-school project education. According to the van Hiele theory discussed in the following section, students have the potential to enhance their cognitive skills and geometric reasoning.

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2.0 THE VAN HIELE THEORY

This study was shaped by the concerns articulated by Dina van Hiele-Geldof (1957/1984), whose aim was to explore how a change in learning methods could enhance educational outcomes. According to Piaget (1975/1985), students' cognitive development depends on their biological maturity. That students' cognitive development depends on the teaching process was argued by Dina van Hiele-Geldof and Pierre van Hiele in their dissertations in 1957 (Fuys, Geddes & Tischler, 1988). Dina van Hiele-Geldof (Fuys, Geddes & Tischler, 1984) in her dissertation had the objective to investigate the improvement of learning performance by a change in the learning method. Central to this model, is the description of the five levels of thought development which are: Level 1 (*Recognition or Visualization*), Level 2 (*Analysis*), Level 3 (*Ordering*), Level 4 (*Deduction*) and Level 5 (*Rigor*). Pierre van Hiele (1986) ultimately defined his model using three levels of thought instead of five: *Visual* (level 1), *Descriptive* (level 2), and *Theoretical* (level 3) (as referenced in Teppo, 1991, p. 210). Battista uses "constructivist constructs such as levels of abstraction to describe students' progression through the van Hiele levels" (Battista, 2011, p.515). He "*has elaborated the original van Hiele levels to carefully trace students' progress in moving from informal intuitive conceptualizations of 2D geometric shapes to the formal property-based conceptual system used by mathematicians*" (Battista, 2007, p.851). He separated each phase in subphases (Battista, 2007). An elaboration of Battista's initial three levels, which are particularly relevant for secondary students, is presented below:

Level 1 (*Visual-Holistic Reasoning*) is separated into sublevel 1.1. (prerecognition) and sublevel 1.2 (recognition). (p.851). "Students identify, describe, and reason about shapes and other geometric configurations according to their appearance as visual wholes, […] they may refer to visual prototypes**.** […]. Orientation on figures may strongly affect students' shape identifications" (Battista, 2007, p.851).

Level 2 (*Analytic-Componential Reasoning*) is separated into sublevel 2.1 (*Visual-informal componential reasoning*), sublevel 2.2 (*Informal and insufficient-formal componential reasoning*) sublevel 2.3 (*Sufficient formal property-based reasoning*). According to Battista "Students [acquire through instruction] a) an increasing ability and inclination to account for the spatial structure of shapes by analyzing their parts and how their parts are related and b) an increasing ability to understand and apply formal geometric concepts in analyzing relationships between parts of shapes". (Battista, 2007, pp.851-852).

Level 3 (*Relational –Inferential Property-Based Reasoning*) into sublevel 3.1 (*Empirical relations*), sublevel 3.2 (*Componential analysis*), sublevel 3.3 *(Logical inference*) and sublevel 3.4 (*Hierarchical shape, classification based on logical inference*). According to Battista "Students explicitly interrelate and make inferences about geometric properties of shapes. […] The verbally-stated properties themselves are interiorized so that they can be meaningfully decomposed, analyzed, and applied to various shapes". (Battista, 2007, pp.852-853)**.**

Researchers have shown that students "often fail in the construction of a geometric configuration which is essential for the solution of the underlying geometric problem" (Schumann & Green, 1994, p.204). This happens because students at the lower levels "identify, […] their appearance as visual wholes" (Battista, 2007, p.851). Van Hiele emphasized the

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importance of students adhering to the following five instructional phases within each level which are briefly the following (Fuys et al., 1984, p.251):

- *Information (inquiry):* Through discussion, the teacher identifies what students already know about a topic and the students become oriented to the new topic.
- *Directed orientation*: Students explore the objects of instruction in carefully structured tasks such as folding, measuring, or constructing. The teacher ensures that students explore specific concepts.
- *Explicitation:* Students describe what they have learned about the topic in their own words. The teacher introduces relevant mathematical terms.
- *Free orientation*: Students apply the relationships they are learning to solve problems and investigate more open-ended tasks.
- *Integration:* Students summarize and integrate what they have learned, developing a new network of objects and relations.

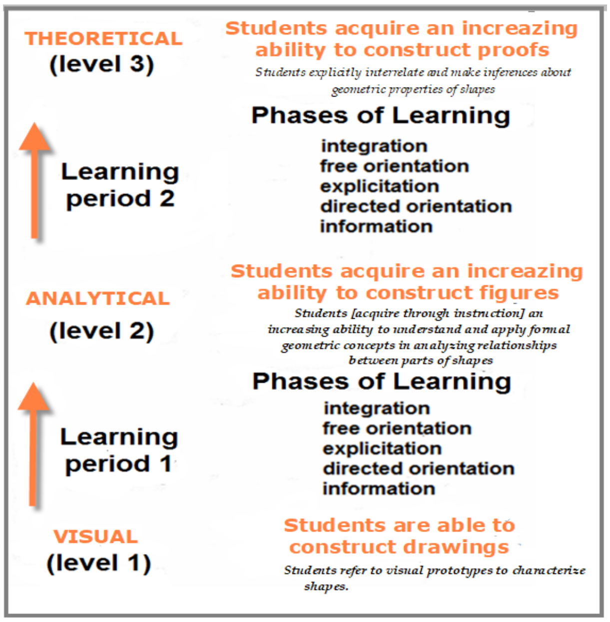

Figure 3. An adaptation on Teppo's diagram (1991, p.210) taking into account Battista's (2007) elaboration of the van Hiele levels (Patsiomitou, 2012a, 2019c p. 120)

Throughout the instructional stages, students advance from lower levels of thinking, characterized by concrete structures, to higher levels, which involve abstract structures. Numerous researchers, including Burger and Shaughnessy (1986), advocate that the sequencing of instruction positively influences students' success. In alignment with van Hiele, Teppo (1991) asserts that each cognitive level is delineated by a learning period during which instruction, structured around the five phases of learning, facilitates students' progression to the subsequent higher level of thought (p.210). The objective of the initial period is to alter students' perceptions of geometric objects (as noted by van Hiele, 1986; Teppo, 1991).

According to van Hiele (1986) "when after some time, the concepts are sufficiently clear, pupils can begin to describe them. […]. The figure becomes the representative of all these properties: It gets what we call the "*symbol character*". In this stage the comprehension of the figure means the knowledge of all these properties as a unity. […]. When the symbol character of many geometric figures has become sufficiently clear to the pupils, the possibility is born that they also get a *signal character*". This means that the symbols can be anticipated. […]. When this orientation has been sufficiently developed, when the figures sufficiently act as signals, then, for the first time geometry can be practiced as a logical topic" (p. 168). This means transforming the visual image or 'drawing' what they perceive, into a 'figure' (e.g., Parsysz, 1988; Laborde, 1993; Mariotti, 1997; Patsiomitou, 2008a, 2012a) with concrete properties. The use of computer software can effectively support the student's progression through van Hiele levels. According to Gawlick (2005, p.370), *a dynamic approach is more suitable for fostering advanced level thinking, as tasks designed for lower levels can be extended to higher levels, thereby encouraging students to develop a habit of 'discovery'*. Additionally, this approach offers a foundational material base for the sequential phases of van Hiele learning, allowing students to investigate the subject during a directed orientation phase and subsequently construct new concepts based on their prior knowledge. In Level 2, students are also beginning to develop the ability to 'construct figures' (Gawlick, 2005, p. 370).

3.0 VERGNAUD'S CONCEPT OF SCHEMA

The study of learning processes is fundamentally connected to the evolution of cognitive psychology, which has primarily concentrated on child development and, more generally, the growth of living beings. The concept of schema (with the plural forms being schemata or schemas) holds significant importance in the field of cognitive psychology. Its origins can be traced back to ancient Greece, where it was considered in a more general context rather than the specific framework utilized in cognitive psychology. According to Corcoran & Hamid (2022):

> A *schema* (plural: *schemata*, or *schemas*), also known as a *scheme* (plural: *schemes*), is a linguistic "template", "frame", or "pattern" together with a rule for using it to specify a potentially infinite multitude of phrases, sentences, or arguments, which are called *instances* of the schema. […]. The Greek word 'schema'; was used in Plato's Academy for "[geometric] figure" and in Aristotle's Lyceum for "[syllogistic] figure". Although Aristotle's syllogistic figures or "schemata" were not schemas in the modern sense, Aristotle's moods were.

The concept of schema holds a central position in Vergnaud's theory of *conceptual fields* (e.g., 1988, 2009). The theory of conceptual fields is the "reference theory" for many authors concerning systems of representation (Hoyles & Noss, 1996; Kaput, 1992). As previously discussed, schemas "coordinate and organize" the observable patterns of a subject's activity with their mental representations. In the words of Vergnaud (1998, p. 172), the concept of schema "*is more general and relates to different kinds of activities*." Examples of schemas for

Vergnaud include telling a story, constructing a political speech, speaking a foreign language fluently with certain specific mistakes and accents, and engaging in dialogue with a group of people. These are what he calls *verbal schemes or social schemes* (Vergnaud, 1998, p. 172).

In Rabardel's publication "People and Technology: a cognitive approach to contemporary instruments" , are identified various types of utilization schemes for a tool (Rabardel, 1995, p.84): (a) *Usage schemes related to "secondary tasks"*: schemes oriented towards managing the tool (b) *Instrumented action schemes* which consist of wholes deriving their meaning from the global action which aims at operating *transformations on the object of activity*: schemes oriented towards executing a specific goal. (c) *Instrument-mediated collective activity schemes* which concern the specification of the types of action or activity, of the types of acceptable results etc. *when the group shares a same instrument or works with a same class of instruments*: schemes concerning the coordination of actions of individual persons as a contribution to the success of common goals. Concerning the notion of *'instrumental schemata'* (Patsiomitou, 2025) it is evident that both students and educators develop/cultivate, foster and enhance ['instrumental'] schemata, indicating that these schemata are dynamic and undergo continuous development within a digital or AI environment (e.g., Patsiomitou, 2024 a, b, c, 2025). These categories of schemas play a crucial role in the field of education, especially within the context of mathematics education. This is due to the fact that when a student encounters a novel mathematical scenario in class, they are able to recall a set of schemas that they have previously developed. These situations result in the creation of novel mathematical schemas or the enhancement of existing ones. For instance, when some students collaborate to solve a problem in a DGS environment, they may develop social, verbal schemas, as well as schemas related to mathematical concepts or instrumental schemata. Students or educators are able to create instrumental schemata throughout the process of *instrumental genesis* (Rabardel, 1995; Trouche, 2003, 2004). The concept of schema is the most important concept in cognitive psychology (Vergnaud, 1998, p. 172). According to Vergnaud, a mental schema consists of four basic components (1998, p.173): (a) *goals and anticipations* to achieve potential objective goals, (b) *rules-of-action* (: the rules that generalize the sequence of actions to be taken, typically in "if... then" forms that contain the operational invariants used by the subject in practice), (c) *operational invariants*, which include the concepts of *theorem-in-action and concept-in-action* that form the cognitive content of schemas, and (d) *inference possibilities* (:the outcomes determined based on the information available to the subject).

The dynamic diagrams' reconfiguration through the complex synthesis of combinations of transformations can lead to a continuous interaction of dynamic figures' *discursive, visual and operational apprehension* (e.g., Patsiomitou, 2008b, c, 2010, 2011a, 2012a, b, 2013, 2014, 2018b). In the words of Dina van Hiele (1984) the diagram goes through a *metamorphosis* as a result of the manipulations of reconfigurations "*followed by a phenomenological analysis and an explicating of its properties: it becomes what we call a [dynamic] geometric symbol*" (Dina van Hiele in Fuys et al., 1984, p.221). The meaning of *'procept-in-action'* (Patsiomitou, 2019b, c) for the DGS environment could thus support the appearance of operational invariants and the development of instrumental schemata during the problem-solving situation as well as influence students' interactions with a dynamic object (Fig. 4). As a dynamic synthesis changes in the linking pages, the (student-) user's cognitive structure undergoes a transformation influenced by their organized instrumental schemata regarding dynamic objects.

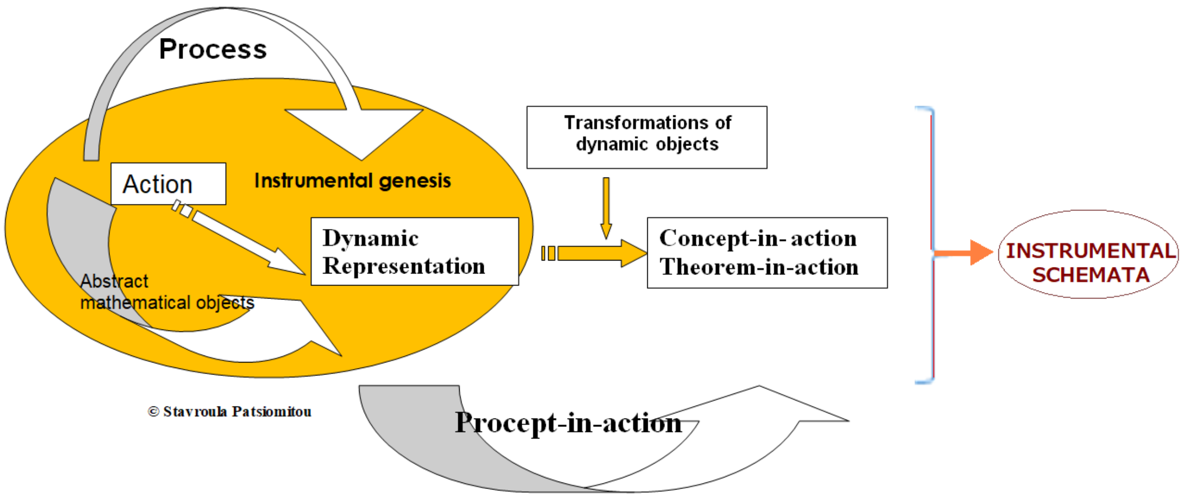

Figure 4. A procept-in-action (Patsiomitou, 2019b, p. 44) during instrumental genesis leading to instrumental schemata (an adaptation for the current study)

In the context of the discussion regarding the process of rediscovery-reinvention of concepts through representational systems, it is necessary to answer certain questions, such as (Vergnaud, 1990): *what is the nature and function of a new concept, a new process, a new type of reasoning, a new representation?* When students encounter a novel situation, they draw upon the knowledge they have acquired from previous, less complex experiences to adapt to the new context (Vergnaud, 1988, p.141). This process is related to Piaget's processes of accommodation and assimilation (e.g., Piaget, 1975/1985). The development of conceptual frameworks in mathematics relies on the activities carried out by students both individually and as part of a group or class, as well as their engagement with other members of the school community [especially with the educator and their classmates].

4.0 DGS: A GENERATOR OF "ALIVE" OBJECTS

In my research titled "*An 'alive' DGS tool for students' cognitive development*" (Patsiomitou, 2018b), I reported the subsequent advantages and effects on students' cognitive processes related to DGS software (e.g., the Geometer's Sketchpad), which were also redefined in Patsiomitou (2019c, p. 74):

- **A first and very important effect on students' thinking** stems from the Sketchpad software allowing the user to create sequential linking pages so that the whole Sketchpad file becomes an "*alive book*" (Patsiomitou, 2005a, p. 63, in Greek; Patsiomitou, 2014, 2018b). The *"alive digital representations*" (Patsiomitou, 2005a, p. 67) function, which makes the whole figural diagram "*alive*", giving the students *the potential to focus their attention on simultaneous modifications (and transformations) of objects on the screen* (Patsiomitou, 2005a, p. 68), also yielded important results during my investigations. According to Sketchpad Help system "Over time, you may want to add additional pages to a document. For example, you may want to organize a series of sketches that develop an argument; you may want to present an activity that has several parts; or you may want to explore a conjecture in more depth than would be possible in a single sketch".

- **A second important effect on students' thinking** stems from the *dynamic transformations in a DGS environment,* a way of modifying an object on screen. We can change a figure's orientation, a figure's size or we can reconfigure it from its parts (Duval, 1995a, b, 1999). Translations, rotations, and reflections are the kind of geometric transformations that preserve the size and shape of a figure. Any transformation (i.e. rotation, translation, reflection) of an object on screen produces a similar or congruent object image on screen. If we drag any point of the object the same transformation occurs to the image object that means that the image object (or reversely) follows the dragging results that refer to the object (e.g., Patsiomitou, 2009 a).
- **A third important effect on students' thinking** occurs from dynamic constructions, that are the constructions created in a DGS environment. Daniel Scher (2002) in his study describes the characteristics of a traditional static construction in contradiction to a dynamic construction. The static constructions possess two characteristics as Scher (2002, p. 1) states: "they are static and particular". In Scher's (2002) words "*the dynamic objects can be moved and reshaped interactively […and] a single on-screen image represents a whole class of geometric objects*" (p.2).
- **A fourth important effect on students' thinking** occurs from the construction of custom tools /scripts (e.g., Patsiomitou, 2005a, b, 2006 a, c, d, e, f, g, 2007a, b, c, e, 2008d, 2009 b, c, 2012a, b, 2014, 2020c, 2021b, 2022a, b, c, d, 2023a, b). As Sträßer (2001) supports:

  "*Apart from practical considerations (like exactness and ease), DGS-use can be structured according to conceptual units by means of macro-constructions. DGS-constructions are not bound to follow the small units of traditional drawing practice. Offering new tools that are unavailable in paper and pencil geometry, DGS-use widens the range of accessible geometrical constructions and solutions. If these tools become everyday instruments in the hands and minds of the user*" (p.332). During the construction of a custom tool a user determines the order the dynamic objects have to be created. This is in accordance with what Balachef & Kaput (1997) support: "*The order in which actions take place could become arbitrary in the eyes of users, which can have significant consequences. [...] This demonstrates the impact of the orientation of the plan which is in general forgotten in elementary geometry, but is recalled to the user as a result of the sequencing of actions (Payan 1992)*". (p.13). I shall further discuss the meaning of custom tools in the next section as instrumental schema generators.

- **The fifth [and most] important effect on student's thinking** stems from the DGS software's *dragging facilities*. Sketchpad's dragging behavior transforms an object on screen moving that object on the screen. According to Laborde (1994, cited in Scher, 2000, p. 43):

  "*The idea of movement in geometry is not new—the Greek geometers devised various instruments to describe mechanically defined curves—but the use of movement was nonetheless 'prohibited in strict geometric reasoning' for reasons that were more metaphysical than scientific. The 17th century marked a break with Greek tradition, and the use of movement to establish a geometric property or carry out a geometric construction became explicit. […] This idea was first expressed in school geometry by the replacement of the geometry of Euclid's Elements by the geometry of transformations* *(which continues to be the only kind of geometry taught in some countries)—quite some time, one must point out, after the characterization of geometry as the study of the invariants of transformation groups, and also quite some years after a daring proposition made in France by Meray (Nouveaux éléments de géométrie, first edition 1874) [...] Meray's idea was to teach geometry through movement: translational movement allowed for the introduction of the notion of parallelism; rotational movement led to perpendicularity*. (pp. 61-62, French original, Scher, 2000, p. 43)

A special report has been prepared regarding the customized tools, that have intrigued me and captured my attention since 2005, when I employed them extensively to create new geometrical and mathematical tools to fulfil my objectives.

5.0 DG CUSTOM TOOLS AS INSTRUMENTAL SCHEMA GENERATORS

The idea of scripting/constructing custom tools was to create "personal tools", or tools that a student could use for his/ her needs. In my view, the subsequent statement could be considered a definition for a custom tool (Patsiomitou, 2008d, Patsiomitou, 2018b, p.51):

Custom tools are 'alive' encapsulated entities developed within a DGS environment, serving as reference points for the organization, retrieval, and reversal of information. This functionality aids in anticipating and manipulating action schemes during the instrumental genesis process. Furthermore, a custom tool can act as a catalyst for students' cognitive growth and the enhancement of their abstract thinking skills.

In the words of Scher (2000, p.45) "*Jackiw viewed the scripting feature of Sketchpad as a way for students to start from the "atoms" and gradually build their own collection of reusable multi-step constructions*". Kadunz (2002) also states that "*to the user, the macro function is a black box producing defined output from defined input*" (p. 74). By constructing a custom tool, we can help students to extend the capacity of their working memory, since the knowledge, the student must retain, is reduced. *Working memory holds only the most recently activated, or conscious, portion of long-term memory, and it moves these activated elements into and out of brief, temporary memory storage* (Dosher, 2003 as referenced in Sternberg et al., 2012).

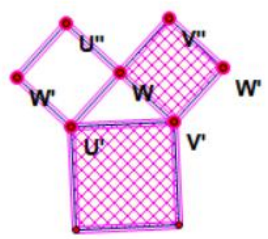	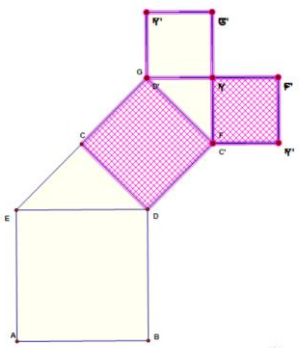	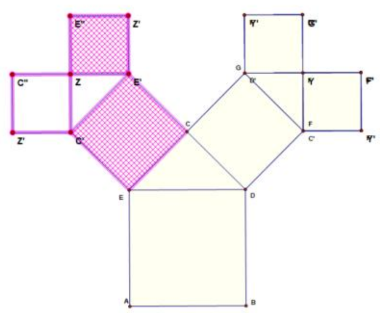
Figure 5a: The initial phase of constructing the Pythagorean Tree Generator-[1] custom tool	**Figure 5b.** The 2nd step of construction involves utilizing the Pythagorean Tree Generator-[1]	**Figure 5c.** The 3rd step of construction

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Nonetheless, the basic underlying notion is that a student is able to codify a construction and the concrete codification shape what the student can do when s/he will encounter a new situation related to the concrete that has been abstracted and codified with the use of custom tool.

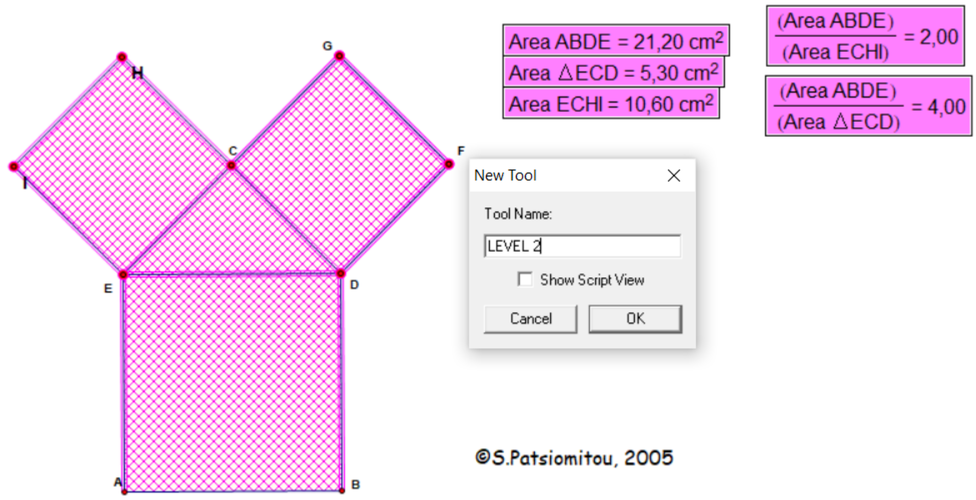

Figure 6a. Creating the Generator- [2] custom tool “LEVEL 2” incorporating measurements and calculations

I shall provide an example to illustrate it. In order to create a Pythagorean tree for each side of the right-angled triangle, it is necessary to develop the Pythagorean Tree Generator- [1] (refer to Fig. 5a) and store it as a custom tool [or create every step from scratch]. This procedure will be reiterated to build the second and third steps, as depicted in the figures above (Fig 5 b, c). The actions we could accomplish are the following: (a) constructing the custom tool “The Pythagorean Tree Generator- [1]” (b) implementing the custom tool to the sides of the triangle (c) constructing a custom tool from the 3rd step of construction of the Pythagorean Tree (Fig.5c), *encapsulating the previous construction* (d) implementing the latest custom tool iterating the process. This way of construction is in a more abstract level than the previous way, as the student is pushed through the process *to a reification of sequential nested objects* (Patsiomitou, 2019c, p. 80). This action has a presupposition: that students are aware beforehand that a side of a triangle is a segment or comprehend the dual function of the objects (van Hiele level 3). Moreover, the orientation of the sides may pose a cognitive obstacle, especially for students at van Hiele levels 1 or 2. This is because students very often fail to recognize the modification of the orientation of tools due to a lack of *place way apprehension* during the *instrumental decoding* process (Patsiomitou, 2011, p.362). The custom tools help them to simplify the construction process. A script /custom tool combines in a concrete and sequential order the steps that have been used to accomplish the construction. For example, if someone construct a square, s/he can save the concrete construction in a custom tool which can repeat the construction in the concrete way used by him/her [the creator of the custom tool], meaning that is processes the objects in the same sequence of the predefined *instrumental trajectory* (Patsiomitou, 2021 a, b). The dragging of the custom tool constructed on screen follows the rules that refer to the primitives and commands incorporated into the custom tool (i.e. if we have measured angles or segments, or calculated a ratio, during our construction of the tool, then the concrete measures and calculations are repeated any time we implement the custom tool). If we drag the tool (Fig. 6a, b), the measures follow the increasing or decreasing of the length of the segments and angles (e.g., Patsiomitou, 2005a, p.83).

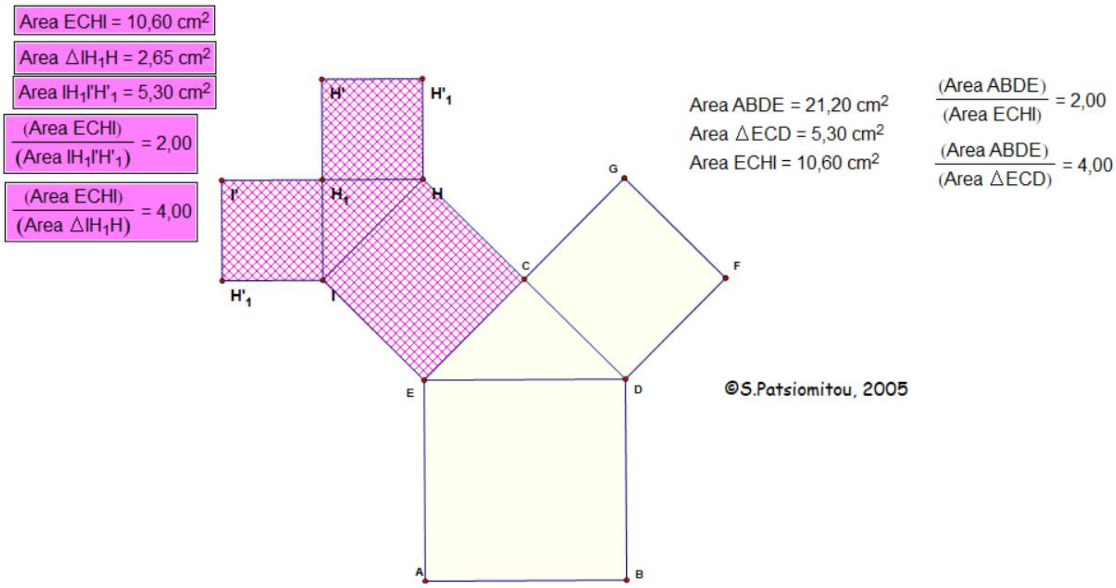

Figure 6b. Implementing the Generator- [2] custom tool

As a result of the construction and application of a custom tool the direct perception of the user is attained with regards to the steps in the development of the construction pertaining to (see) (e.g., Patsiomitou, 2007a, 2014, 2018a, b, 2019a, c): 1) the repetitions in the measurements or calculations of the areas of initial shapes 2) the developmental way of the construction of the figure and 3) its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file. If we have constructed a custom tool which incorporates the use of iteration processes, in the case of Geometer's Sketchpad the application of the custom tool will include the iteration at every new step during every new application of the custom tool. Mariotti (2000) declares that in a construction generated using dynamic geometry software "[…] *the elements of a figure are related in a hierarchy of properties, and this hierarchy corresponds to a relationship of logic conditionality*" (p.27). This is in accordance with what Jones (2000) points out that "*dynamic geometry systems (DGS) would seem to have the potential to provide students with direct experience of geometrical theory, and thereby break down what can be an unfortunate separation between geometrical construction and deduction*" (p.56).

6.0 PYTHAGOREAN TREES AS FRACTAL CONSTRUCTIONS

The dynamic geometry software Geometer's Sketchpad includes the "Samples" folder, which contains the "Geometry" subfolder, within which is the software's fractal's gallery. On the first page of the multi-page file of the fractal gallery, the dynamic Pythagorean tree appears. In this context, the point that triggers the movement of the branches of the Pythagorean tree is located at the vertex opposite the hypotenuse. As this point moves, the branches of the tree rotate,

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generating new, smaller branches in accordance with the fractal construction process. This motion represents the reproduction of the original structure of the Pythagorean tree at a reduced scale with each iteration, and the process continues to unfold as a fractal structure. The details regarding the construction of the Pythagorean tree were presented at the PRISM II conference, focusing on *Construction and Iteration*: *mathematical generalization in Dynamic Geometry*. As it is reported: *To iterate an action is to repeat it some number of times. In geometry, an iteration uses an operation performed on one set of geometric objects to produce a new set of objects that share the same relationship. You've just constructed squares on the three sides of a right triangle. You're now going to iterate this right-triangle construction to produce a tree-like construction whose branches contain smaller and smaller copies of your triangle and three squares.* For further information, please refer to the description of the Pythagorean tree construction available at the following link:

https://learningcenter.dynamicgeometry.com/PythagoreanTheorem.htm.

Figure 7 below is incorporated in my book (Patsiomitou, 2009b, 2022a), where I detail the sequential *instrumental trajectory* (Patsiomitou, 2021a, b), I used for the construction, using *linking visual active representations*. My primary goal was to investigate the ways in which the software assists students in creating patterns associated with the repetition of objects during the construction process, the formulation of measurements and calculations, and the positioning or reorientation of shapes on the plane.

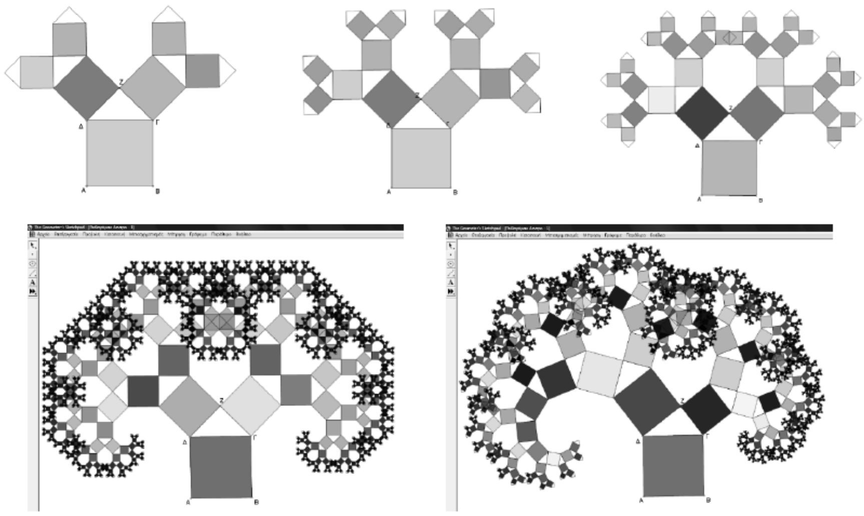

Figure 7. Linking the sequential phases of constructing the Pythagorean Tree (Patsiomitou, 2009b, 2022a)

Furthermore, I examined the *generalization of the process*, considering the dynamic relationship between the fractal construction and the measurements/calculations in a linked table, along with its connection to graphical representations. Ghosh (2016) asserts that "*One of the foundational aspects of developing algebraic thinking is the ability to generalize. Research describes two kinds of generalization (Kinach, 2014), namely, generalization by analogy and generalization by extension. Generalization by analogy refers to observing a* *pattern, extending a sequence to the next few terms and being able to relate a particular term of the sequence to its previous terms. This kind of generalization requires recursive thinking. Generalization by extension, on the other hand, refers to writing a formula for the nth term of a sequence – which requires explicit thinking". (p.59).* The values in the table are updated when the fractal object is moved, allowing us to visualize them in a graphical representation. In the following diagram, the interconnection between (a) a fractal Pythagorean tree (shape), (b) the table of measurements and calculations, (c) the graphical representation of the sequential areas, where the pre-designed sliders allow control of the epsilon-delta definition, the limit of the sequence, and (d) the symbolic formula of the definition, is shown. Through the activity, the students are called to assemble the components of the definition and approach as closely as possible the rigorous definition of the limit of a sequence (Fig. 8).

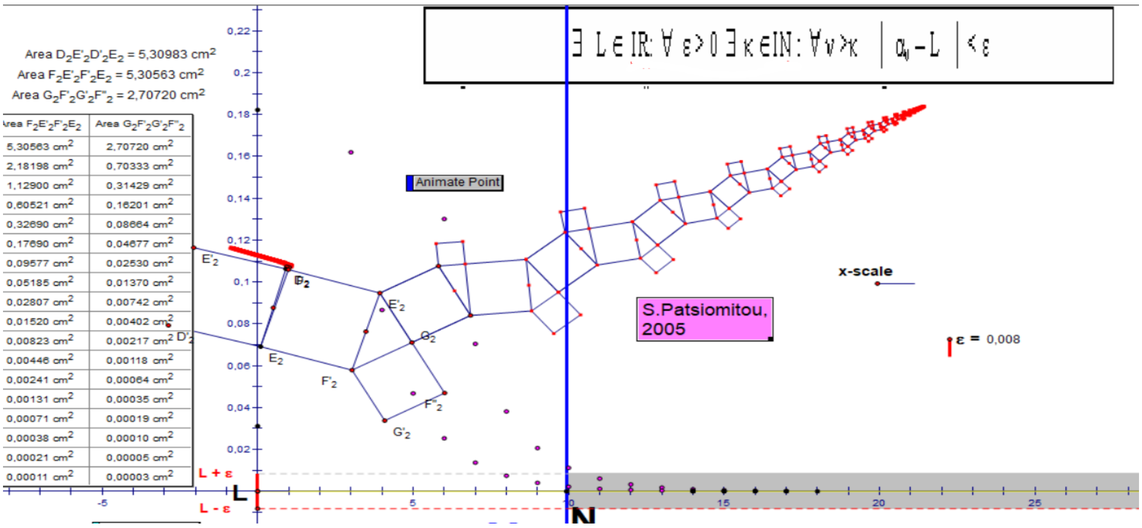

Figure 8. Linking Visual Active Representations using a Pythagorean tree structure (Patsiomitou, 2005a, pp. 81-82, 2007a, 2014)

If we use the visual representation of the proof of the Pythagorean theorem to repeat the construction process on each branch, we will arrive at a fractal construction of the Pythagorean tree. The structure of the figure sequentially and in reduction remains the same as the construction continues. Thus, repetitions of the original structure are created, branching Pythagorean visual proofs, which, however, continuously lead to ever smaller figures, triangles, and squares. Therefore, the sizes of sides, perimeters, areas, angles, arcs, etc., continuously decrease. What is the final value of the area of the square at the end of each branch? What does this area tend to? What is the ratio of two consecutive measurements? For the needs of the research process [for the branches of a Pythagorean tree], I created an appropriate custom tool which used the properties of the transform-iteration command (Patsiomitou, 2005a, 2007a).

7.0 THE PYTHAGOREAN TREE CONSTRUCTION USING THE ITERATION COMMAND

I will begin by describing the Generator- [3] custom tool, which is represented by an irregular pentagon (Fig. 9a). We create a square labeled ABCD and identify the midpoint E on side CD.

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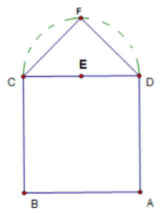

Figure 9a. The Generator- [3] custom tool

Using E as the center and a radius that is half the length of the square's side, we draw a semicircle extending outside the square. By selecting any point on this semicircle and connecting it to points C and D of the square, a right triangle is formed. If we specifically choose the midpoint of the semicircle, the resulting triangle will be both right-angled and isosceles.

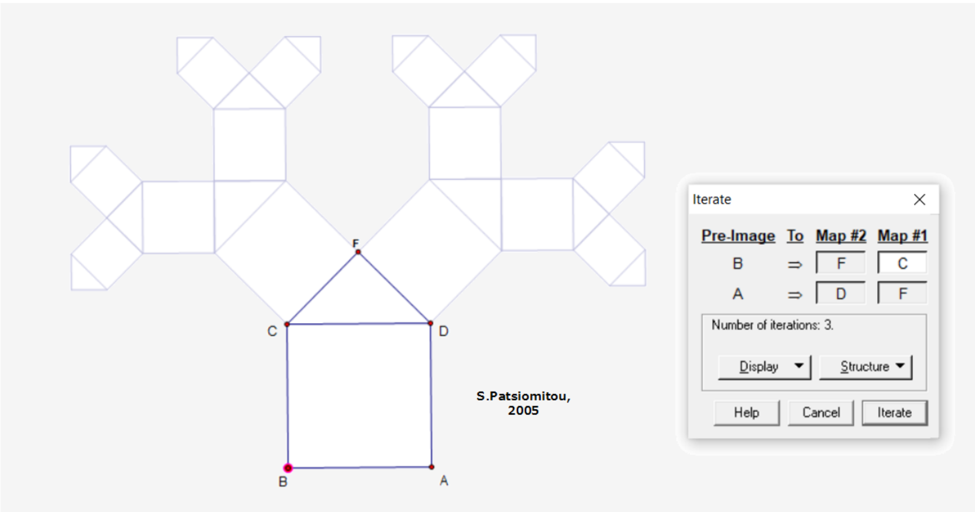

Figure 9b. Implementing the iteration process to the Generator- [3] configuration

The angle CFD measures 90 degrees, as it is an inscribed angle that subtends a semicircle. [Theorem1: *An inscribed angle subtends a semicircle* if and only if the angle is a *right angle* and Theorem 2: *The measure of an inscribed angle is half the measure of its intercepted arc*]. To create a Pythagorean tree, we will repeat this process, but instead of starting with segment AB we will begin with segment CF. This process will produce another smaller square "to the left." Then, we will repeat it again, this time starting from ED, to construct another smaller square "to the right". What is the relationship between the lengths of the sides of successive squares? The lengths of these sides form a geometric progression, decreased by a constant ratio, the ratio of one of the congruent sides of the isosceles right triangle to the hypotenuse (since the legs of the isosceles right triangle are in ratio of $1{:}\sqrt{2}$ to the hypotenuse). Therefore, each successive square has a side length equal to $1{:}\sqrt{2}$ times that of the previous one. Throughout the iterative construction process, it becomes evident that all the resulting triangles possess both isosceles and right-angled properties. This iterative pattern reveals a consistent mathematical relationship among the triangles, squares, and other geometrical elements involved. In particular, when we calculate the lengths of the sides of these shapes, we observe that they decrease according to a geometric sequence, with each subsequent side being reduced by a factor of $1:\sqrt{2}$ in relation to its previous one. So, if the first square's side is equal to a then the next square's sides are: a, $a:\sqrt{2}$, a:2, a: $(2\sqrt{2})$…

Table 1		
Iteration (n)	The length of a side of square Sn	The area of a square E_n
0 (initial)	S_0=a	$E_0= a^2$
1	$S_1=a:\sqrt{2}$	$E_1= E_0:2$
2	S_2=a :2	$E_2= E_1:2$
3	$S_3=a : 2\sqrt{2}$	…
4	S_4= a :4	
…	…	…
n	$S_n= a : (\sqrt{2})^n$	$E_n= E_{n-1}:2= a^2:2^n$

The construction of the Pythagorean tree will be displayed, along with the measurements and calculations for the first n^{th} iterations. Upon examining the properties of the triangles, it is evident that they consistently maintain their isosceles and right-angled characteristics throughout the iterative process. The lengths of their sides decrease in size progressively with each iteration, adhering to a defined mathematical sequence.

Regarding the measurements and calculations, the dimensions of the figures determine a geometric progression, with each subsequent square's side length being reduced by a specific ratio compared to its previous one. The hypotenuse of every isosceles and right-angled triangle is equal to the side of the square that has been produced. Moreover, the perpendicular sides of the triangles are equal with the sides of the squares that are generated during the iterating process.

As it is obvious if the side DC =a then FC=FD= $a:\sqrt{2}$. This observation has to do with the successive square side lengths. What about the 20th iteration? The general formula for the side length of the square at the n^{th} iteration is: $Sn=a:(\sqrt{2})^n$. For the 20^{th} iteration is $S_{20}=a:(\sqrt{2})^{20}$= a :1024. From the calculations shown in the table, we can observe: (a) The side lengths of the squares decrease progressively while maintaining a specific pattern. (b) The relationship between successive side lengths follows a fixed geometric ratio. This decreasing follows the above-mentioned rule, enabling us to predict the side length at any given iteration. This confirms the fractal-like structure of the Pythagorean tree, where each new square is reduced by a predictable factor. With regard to the areas of the successive squares we can prove that: if $n\to\infty$ then

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{a^2}{2^n} = 0$$

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8.0 AN INNOVATIVE APPROACH TO THE PYTHAGOREAN TREE IN EDUCATIONAL CONTEXTS

To initiate the iteration process without employing Generator tools, we start by creating a square, and on one of its sides, we place a right isosceles triangle. By iterating this initial shape seven times, we obtain the configuration illustrated in the diagram, where the active representations facilitate the counting of tree branches (see Fig. 10a).

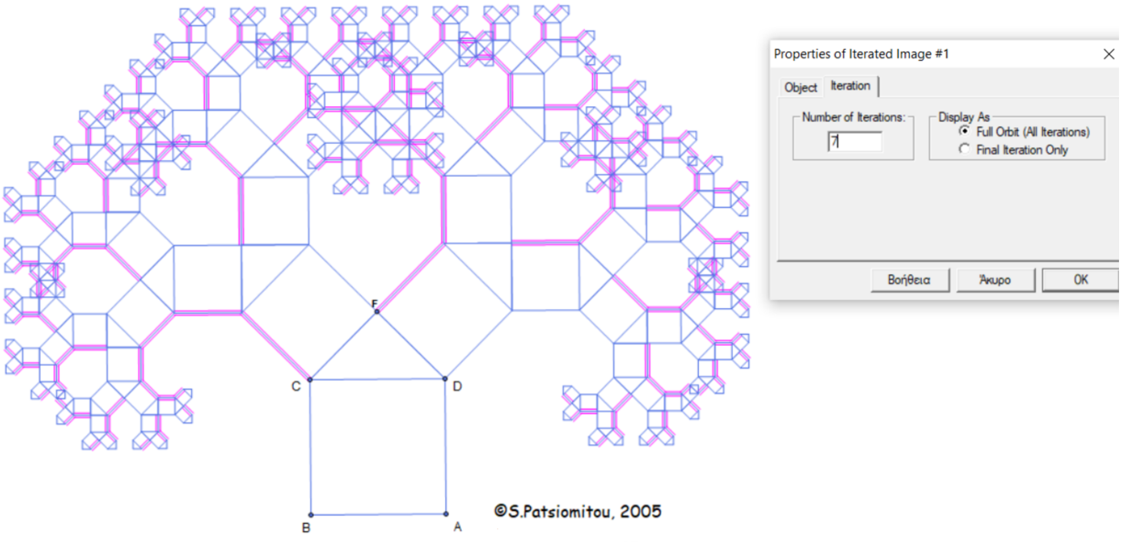
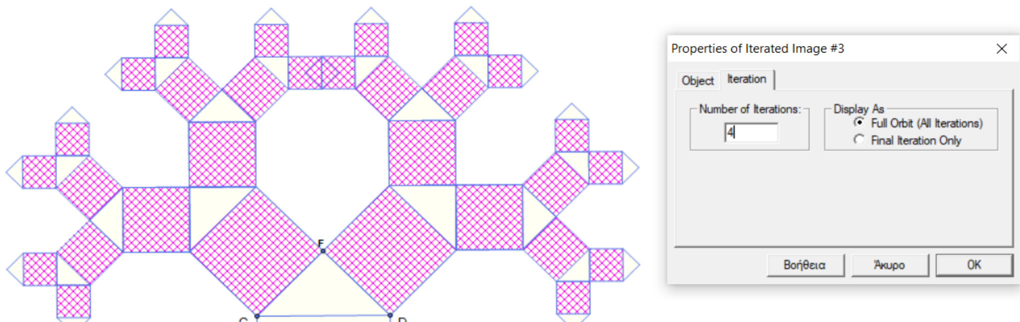


Figure 10a. Active representations facilitating the counting of the tree branches



n	Area ABCD	Area △FCD	(Area ABCD) / (Area △FCD)
0	11,47 cm²	2,87 cm²	4,00
1	5,74 cm²	1,43 cm²	4,00
2	2,87 cm²	0,72 cm²	4,00
3	1,43 cm²	0,36 cm²	4,00
4	0,72 cm²	0,18 cm²	4,00

Figure 10b. Generation of tabularized measurements and calculations by utilizing the iteration process

When we calculate the area of the square and the area of the right triangle, as well as the ratio of their areas, we observe that the measurements and calculations are repeated and tabulated (Fig10b). This repetition aids in understanding the sequence of terms as a decreasing geometric

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progression. We know that if the area of the square is a^2, then the area of the right and isosceles triangle is a^2:4. Therefore, the total area of the Generator- [3] configuration is $E_1 = (5a^2):4$. The perimeter of the square is 4a. Therefore, the length of the vertical side of the right-angled triangle is $x = a\sqrt{2}$. This implies that the total perimeter of the irregular pentagon (: Generator- [3] configuration) is $P_1 = 3a + 2a\sqrt{2}$.

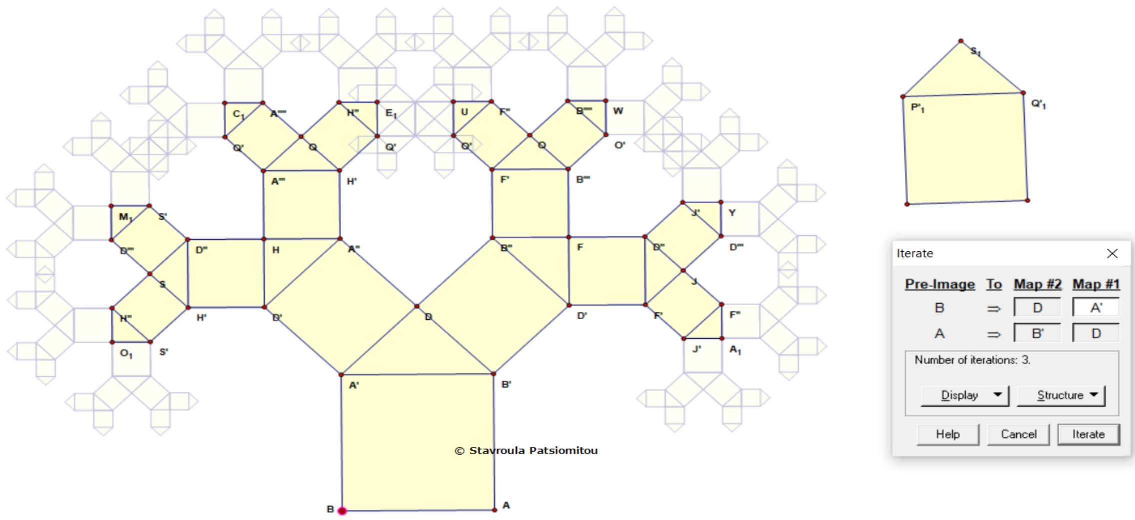

Figure 10c. Implementing the iteration process two times

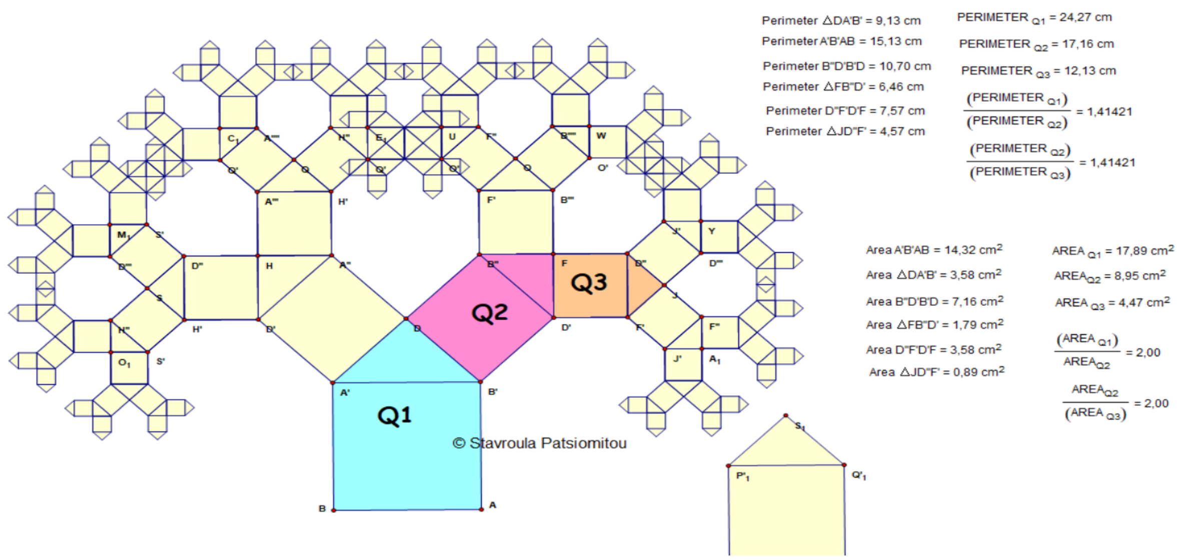

Figure 10d. The visualization of the repetition of the Generator- [3] irregular pentagon

In the figure 10d, we observe the appearance of three distinct sizes of the Generator- [3] irregular pentagon, which warrants further investigation. Specifically, we must consider: What are the reasons for their similarity? What is the ratio of their similarity? Is this ratio repeated?

The iterative process utilized in the figure, leveraging the software's capabilities, produces a figure (Fig. 11a) where various structural shape units appear, such as irregular octagons [and heptagons or other polygons] that partially intersect/overlap one another. The configurations of irregular octagons, heptagons, and polygons in general, as depicted in the illustration (Fig. 11a, b), arise from the visualized *empty spaces* [or negative spaces]— spaces which many children find challenging to perceive as figures. This challenge occurs because such shapes often constitute sub-figures (i.e., components of larger shapes) or overlap with one another, among the branches of the tree. It is essential for the student to direct their attention to one of these shapes, and gradually they may begin to identify all the smaller, similar polygons of reduced scale that emerge through the recursive process. Additionally, is there a discernible pattern in the distribution of these irregular octagons across the plane? In these illustrations, we observe the repetition of empty spaces [/negative spaces] shaped like irregular heptagons and cardioids, which also follow a certain regularity that warrants further examination). As the figure expands, these units continuously decrease in size while increasing in number. This is a subject that requires further investigation: How does their number increase? What type of sequence regulates their development? The orientation of figures can significantly influence students' identification of shapes. Throughout the process, there is an increasing capacity and tendency to consider and analyze the spatial configuration of shapes by examining their individual components and the relationships between those components, along with an improved, enhanced ability to comprehend and apply formal geometric principles in analyzing and evaluating the interconnections and the interrelations of figures' properties.

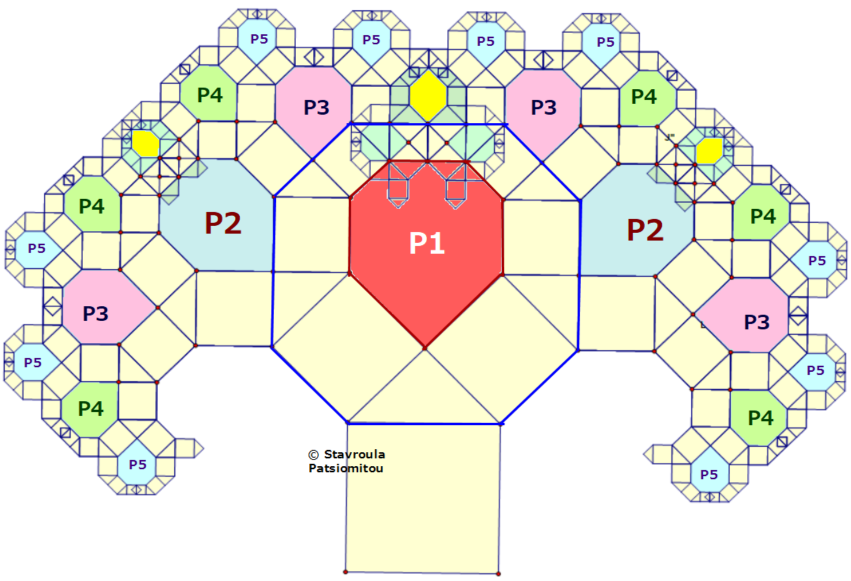

Figure 11a. Irregular heptagons and octagons formed within the empty spaces between the branches

The screenshot (Fig. 11a) from Geometer's Sketchpad illustrates the Pythagorean Tree, which has been generated through iterations utilizing Generator-3. We will now analyse the properties of the figures[/polygons] that are formed in the empty/negative spaces between the branches. We observe the repetition of: *(a) Irregular heptagons of cardioid shape (b) Irregular octagons (c) Irregular hexagons.*

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- **Irregular heptagons of cardioid shape**: $1P_1$, $2P_2$, $4P_3$, $6P_4$, $10P_5$ (+2 overlapped) ... as we observe, follow a concrete pattern, since each term results from the sum of the previous terms: 2 + 4 = 6, 4+6 = 10. Therefore, the next term will be 6 + 10 = 16. If $a_1=1$, $a_2=2$, $a_3=4$, $a_4=6$, $a_5=10$, $a_6=16$ …. then $a_n=a_{n-1} + a_{n-2}$

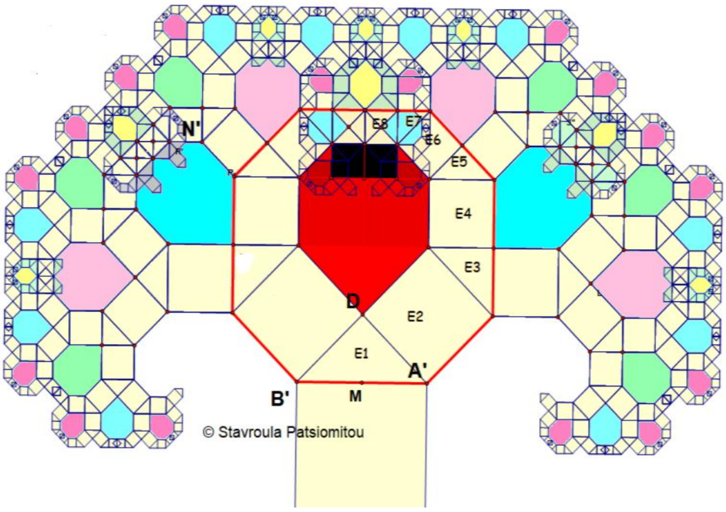

Figure 11b. Irregular heptagons and octagons formed within the empty spaces between the branches

- **Irregular octagons** that contain the heptagons and partially overlap each other. The outlined octagons possess specific properties. In general, the initial observations that emerge relate to their fundamental geometric characteristics: they are all similar to each other, and the ratio of their similarity is determined by the calculations of their perimeters. Indeed, we observe that their similarity ratio is 1.41, which is equivalent to the square root of 2 ($\sqrt{2}$). As per the theorem which asserts that *'the ratio of the areas of similar polygons is equivalent to the square of their similarity ratio,'* the ratio of their areas is $(\sqrt{2})^2 = 2$.

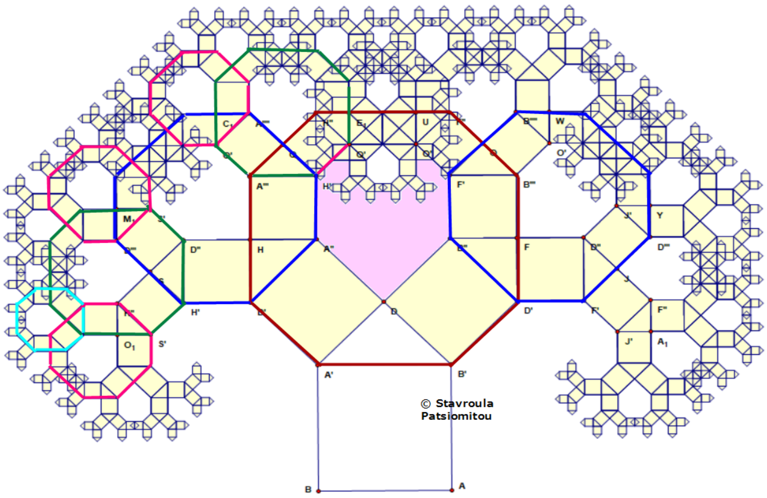

Figure 12. Irregular octagons outlined within the Fractal Pythagorean Tree

The octagons follow a regular pattern as they decrease in size and are reiterated within the figure: (a) a central octagon is partially overlapped by two smaller octagons on each side. (b) four smaller ones are formed, followed by six even smaller ones, and so on. In this way, a sequence of terms emerges: 1, 2, 4, 6, 10... and each term is the sum of the two previous terms for the terms a_3, a_4, etc. That is: 6 = 4 + 2, 10 = 4 + 6… Therefore, the next term will be 16. Thus, the terms are as follows: 1, 2, 4, 6, 10, 16, 26, 42…………

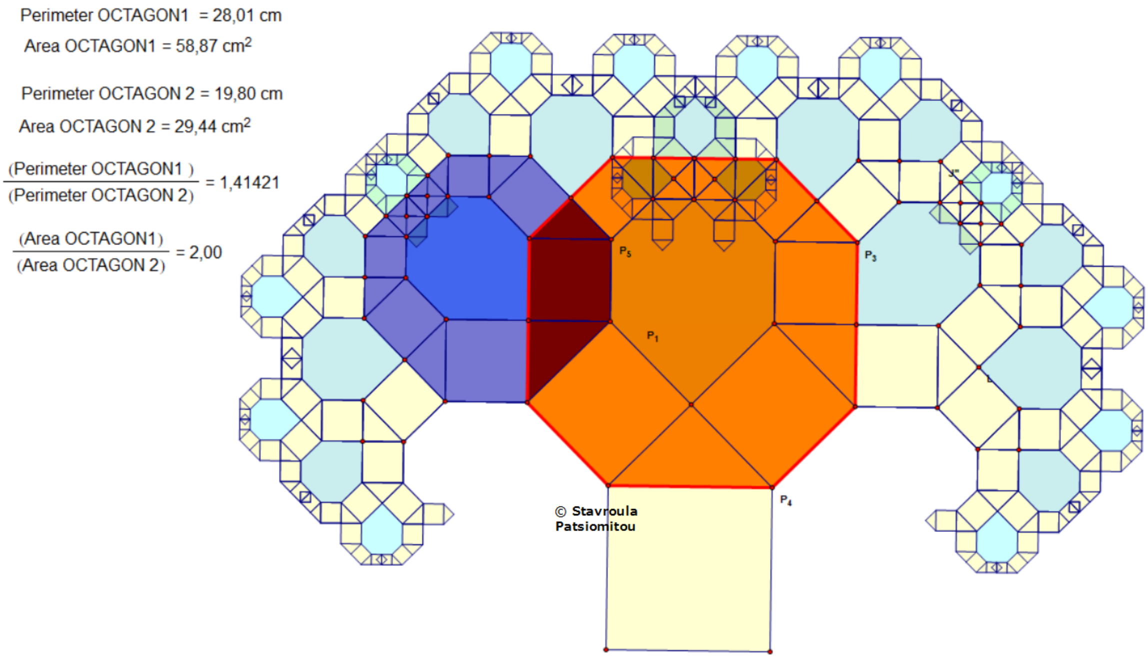

Figure 13. Concentrating on the figures and subfigures of irregular octagons

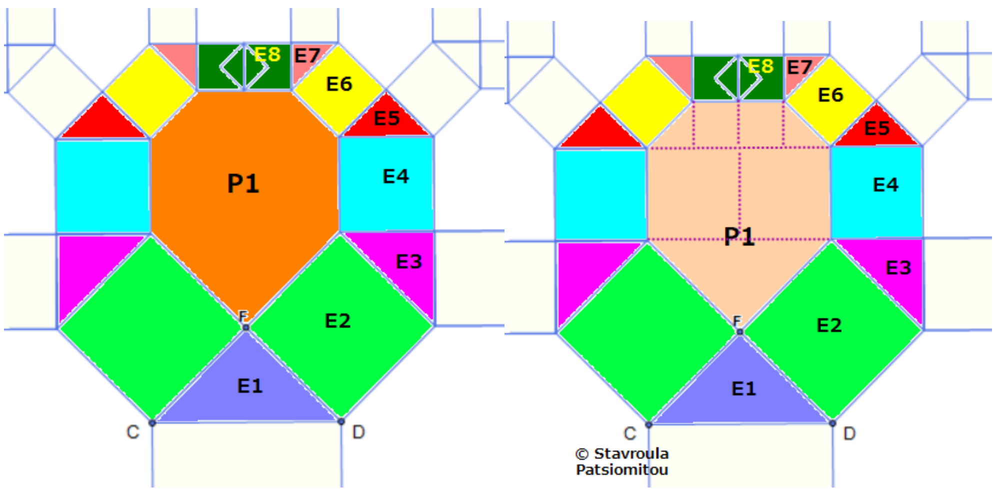

Figure 14 a, b. Concentrating on the characteristics of irregular octagons

The octagon consists of 3 repetitions of successive Generators-[3], which can be easily calculated if we initially subtract the area of the inner irregular heptagon. If ‘a’ is the side of the initial square, then its area is E= a^2 and the area of the right-angled triangle is E1=E/4.

Similarly, $E_2= 2E_1= 2 * E/4 = E/2$, $E_3= E_2/4 = E/8$, Similarly, $E_4= E_2/2 = E/4$, $E_5 = E_4/4= E/16$ και $E_6= 2*E_5= E/8$, $E_7= E_6/4= E/32$ και $E_8= 2E_7= E/16$. Consequently, we reach the subsequent sequence of terms (Table 2).

Table 2	
The area of sequential squares	The area of sequential right and isosceles triangles
E	$E_1= E:4$
$E_2=E:2$	$E_3= E:8$
$E_4=E:4$	$E_5= E:16$
$E_6=E:8$	$E_7= E:32$
$E_8= E:16$	
$\Sigma= E/2+E/4+E/8 +E/16 = 15E/16$	$\Sigma= E/8 + E/16 + E/32 = 7E/32$

The areas of the rectangles and isosceles triangles form a geometric progression with a ratio l=1:2 as do the successive areas of the squares. The heptagon P_1 inside the octagon consists of the following areas $P_1= E_1+ 2E_4+3E_8 = E/4 + 2 E/4 +3 E/16= 15E/16$. Therefore, the total area of the octagon is equal to $\frac{41E}{16} + \frac{15E}{16} = \frac{56E}{16}$.

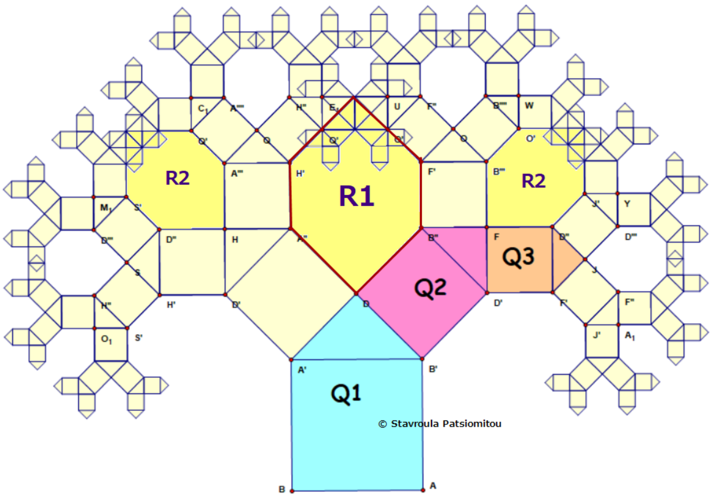

Figure 15. Concentrating on the properties of the irregular hexagons

A closer look at the figure 15 reveals the repetition of **irregular hexagons** symmetrically positioned within the overall shape, each with specific side lengths, perimeters, and areas. For instance, the irregular hexagon R_1 is composed of subfigures whose areas, perimeters can be easily determined.

9.0 ADDITIONAL GEOMETRIC PROPERTIES OF THE PYTHAGOREAN TREE

The Pythagorean tree possesses numerous geometric properties that have been examined by many researchers and teachers who design classroom activities to attract the interest of their students. In the subsequent discussion, I shall present several of these properties that are of interest for developing proof procedures. It is of particular interest if these properties also can be extended when the figure of the right triangle is scalene (i.e., all sides are unequal).

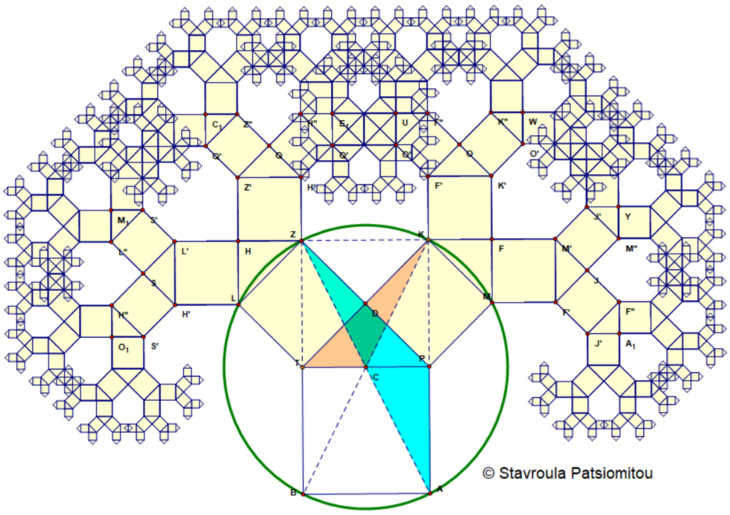

Figure 16. A circle centered at point C that intersects the vertices of the Pythagorean Tree Generator.

We observe that the triangle KCT is congruent to triangle ZCP (since ZP=KT, TC=CP, <T=<P= 45^{o}). Thus, ZC=CK. Similarly, BC=CA. Likewise, ZA=KB (triangle ZAP= triangle KTB since they have ZP=KT, PA=TB, <ZPA=<KTB=135^{o}). Points Z, C and A are collinear (: *Three or more points are said to be collinear if they all lie on the same straight line*) because quadrilateral ZKAB is a rectangle, hence its diagonals bisect each other at C. Therefore, there is a circle with center C passing through the vertices Z, K, A and B. Similarly, it can be shown that the circle also passes through the other vertices L, and M (Fig. 16).

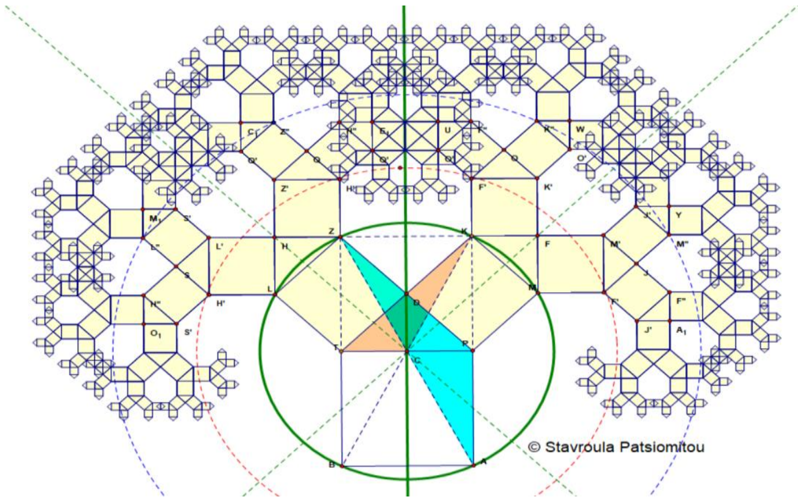

Figure 17. Concentric circles and figures’ axes of symmetry

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A central axis of symmetry can be recognized, which pertains to the central portion of the tree [in addition to two more for the specific part of the tree located at the branches extending to the right and left of the central axis]. Moreover, besides the initial circle, which has been shown to intersect all the vertices of the original figure, it can be easily demonstrated that the iterative process results in the creation of additional concentric circles that also intersect the vertices of the squares formed during the construction (Fig. 17). Furthermore, the axes of symmetry adhere to the same principle [as previously stated] as the branches of the figure are expanded (1, 2, 4, 6 etc.).

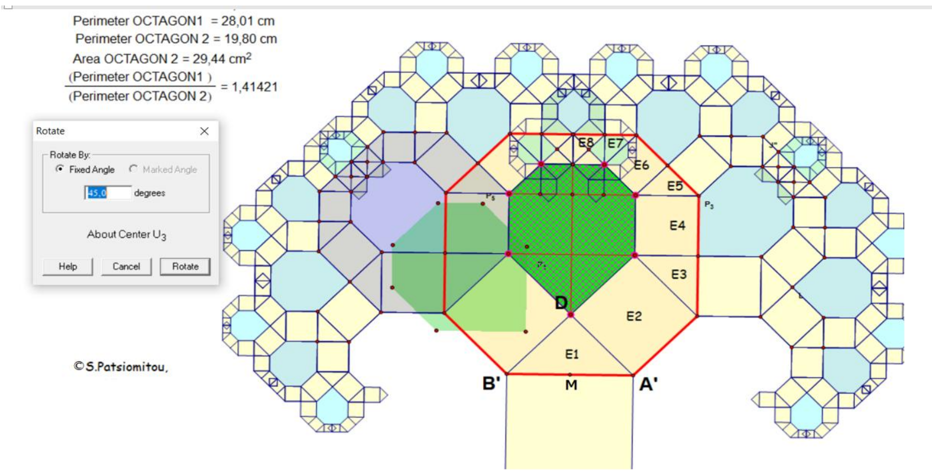

Figure 18a. Rotating the irregular heptagon for 45 degrees by center M

Of particular interest is the presence of transformations in the overall figure (rotations, reductions or enlargements under scale, reflections) which we will briefly review below (Fig. 18a, b, c). The arrangement of the heptagons within the figure is something that gained my attention as it is of particular interest. A rotation of the central heptagon by 45° produces a mirrored image of the figure above, while a rotation by -45° generates the mirrored form below. Notably, the lower image undergoes a reduction by a similarity ratio of 0.5, while the upper image is enlarged by a similarity ratio of 2.

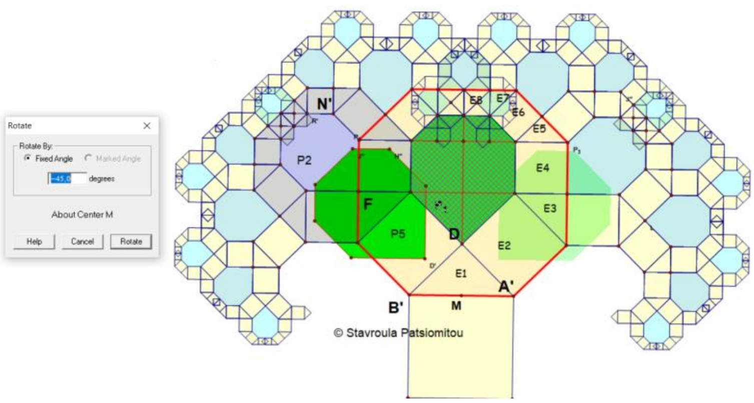

Figure 18b. Rotating the irregular heptagon by center M for -45 degrees

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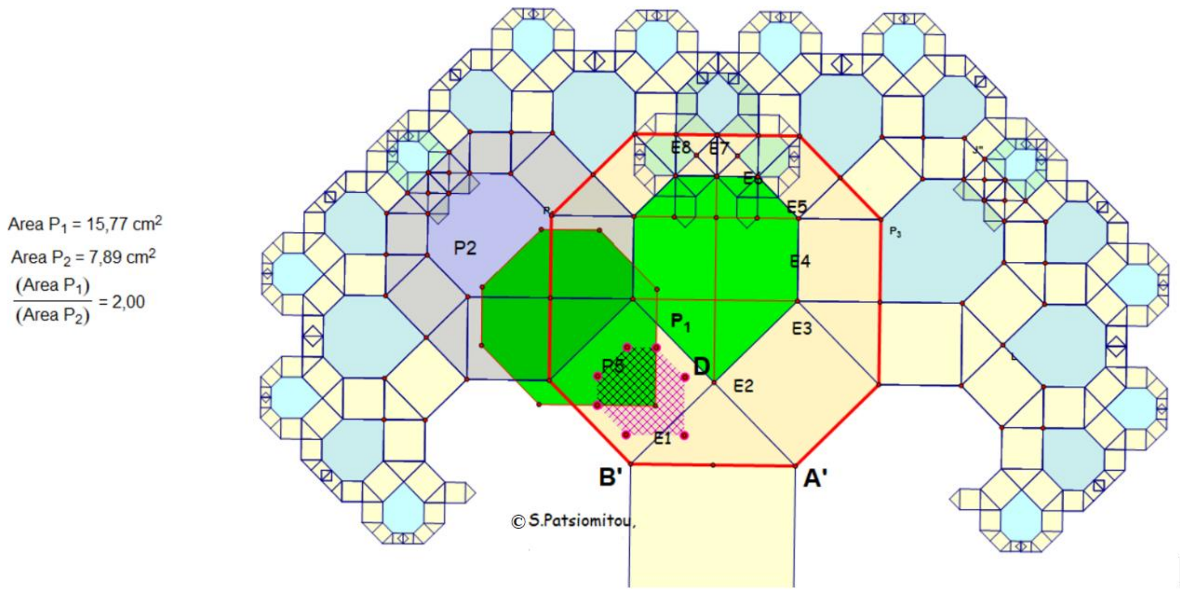

Figure 18c. Dilating the irregular heptagon

Gagné, Briggs & Wager (1992) proposed a systematic instructional design process "*The Events of Instruction and their relation to processes of learning*", following a behaviorist approach for the learning process. Even though I was a constructivist teacher and researcher, I find the "*gaining attention*" principle to be relevant to every moment of my didactic life. In class, there is nothing more important than gaining the attention of the students who think that Geometry is hard and not "nice". By giving them "beautiful mathematics" to construct, I used to "gain" their attention for what follows: constructing meanings. The instructional design process was designed in phases: what I did towards preparing the lesson before the instruction was delivered; what the organized topics were of the learning trajectory; what I predicted regarding the external stimulation delivered by new representational infrastructures in order to create successive stages in the transformation of previously learned material retrieved from the learner's memory etc. To design instruction, I had to establish a rationale for what has to be learned in order to be successful. Concluding, students can improve their cognitive abilities, as described in the van Hiele theory by constructing, discussing, and utilizing the Pythagorean tree structure within instrumental schemata, which will act as case studies for analysis in this research study.

10.0 A FRACTAL-BASED DYNAMIC PROGRAM FOR MATHEMATICS EDUCATION: FROM ZERO TO INFINITY

This section introduces the innovative Fractal-based Dynamic Program (FDP): "*Fractals – From Zero to Infinity"* (Patsiomitou, 2016 a, b, in Greek) which has the potential to serve as an informal curriculum cantered around the concept of fractals. It aims to incorporate fractal geometry into secondary education by utilizing dynamic mathematical tools. The FDP covers a range of mathematical subjects, such as geometry, algebra, pre-calculus, through the exploration of fractals, focusing on, experiential learning activities, digital interactive resources, and problem-solving tasks. I had submitted my proposal with regard to FDP for approval to the [Greek] Governing Committee of Model and Experimental Schools (D.E.P.P.S.) the school years 2011-12, 2012-13 and 2013-14. The program consisted two hours of lessons each week, scheduled after school. Instruction took place in a classroom that featured an interactive whiteboard, enabling students to engage in small group experiments while sharing the computer mouse. The following discussion of FDP is a brief presentation of the unofficial curriculum, concerning my proposal approved by D.E.P.P.S., and accompanied by a brief summary of the Program I designed and implemented. The objective of the Program "Fractals – From Zero to Infinity" was to offer students the chance to experiment, investigate, comprehend, and relate mathematical concepts they had previously encountered in the mathematics classroom or through online resources. The program offered a chance for a deeper exploration and an enjoyable approach through interactive activities aimed at cultivating a passion for mathematics in students. Furthermore, students participated in the construction of fractal objects using traditional paper-and-pencil methods, acknowledging the limitations of static media and traditional geometric tools for creating fractals (realizing also the inadequacy of traditional geometry tools for creating these objects). They frequently compared the geometric tools they used with the software tools, recognizing the capabilities and limitations of each. Subsequently, students became familiar with the concepts of transformation (e.g., rotation, reflection, translation by vector, and iteration). They examined the similarities and differences between these concepts and the concepts of central and axial symmetry as outlined in their class curriculum. In this context, they carried out transformation and fractal constructions using the dynamic geometry software Geometer’s Sketchpad, learning both the software tools and the theoretical concepts simultaneously. The subsequent units aim to present fundamental mathematical concepts through the lens of fractals, highlighting self-similarity, iterative processes, and the concept of zero and infinity. I developed the FDP Program units in six months. As I already mentioned, the instructional resources utilized in the FDP Program incorporate several chapters from my book, *Learning Mathematics with Geometer’s Sketchpad v4* (Patsiomitou, 2009 a, b), updated in 2022 under the title *Conceptual and instrumental trajectories using linking visual active representations created with the Geometer’s Sketchpad* (Patsiomitou, 2022a). This book integrates and elaborates on my earlier research, which I have presented at conferences in Greece and published in academic journals. Additionally, I acted as the instructor for the FDP Program for three years, during which there were slight inconsistencies/ discrepancies in the content; nevertheless, the results concerning the students' learning were exceptional. Specifically, the constructions and explorations through the Geometer’s Sketchpad DGS environment included the following units:

Unit 1: The Road to Infinity – From Zeno of Elea to Benoit Mandelbrot (*An Introductory Historical Overview*).

This section represents the *Information (inquiry) phase* of the FDP Program. It serves as an introductory unit that examines the historical development of fractals, following the concept from ancient Greek philosophy to contemporary mathematics. The Unit 1 incorporates and presents the essential characteristics and fundamental properties of fractals, including self-similarity, and investigates their presence in the natural world. Through the study of mathematical fractals like the Sierpinski Triangle, Sierpinski Carpet, Menger Sponge, and Koch Snowflake, students acquire a comprehensive understanding of fractal theory and its infinite characteristics. They learn about important mathematicians like Mandelbrot, Georg Cantor, and Konstantinos Karatheodory. This unit also introduces students to the concept of fractional dimensions (e.g., Löfstedt, 2008) and their implications for traditional Euclidean geometry. The primary components of this unit are outlined in the subsequent research questions and procedures.

- What are fractals? What are the defining properties of fractal objects?
- What is self-similarity? In what ways does self-similarity appear in the natural world? Examples and illustrations of natural fractals.
- The idea of dimension within Euclidean geometry — The concept offractal) dimension.
- An overview and introduction to essential mathematical fractals, including the Sierpinski Triangle (also known as the Sierpinski Sieve), the Sierpinski Carpet, the Menger Sponge, the Koch Snowflake, as well as the Julia-Mandelbrot sets. Fractal constructions on the plane using traditional or dynamic geometry tools.

Unit 2: Tessellations of the Plane

This section denotes the *Directed orientation phase* of the FDP Program, as per the phases outlined in van Hiele's theory (1986). Students interact with instructional materials through meticulously -very thoroughly- designed activities, including folding, measuring, and constructing. This phase facilitates the exploration of particular concepts, with tessellations, or plane tilings, being a fundamental focus in the study of geometry. This unit introduces the concept of regular polygons, inscribed and circumscribed within a circle. Students explore the properties of these polygons and their relationship with the number π (e.g., Patsiomitou, 2006f, 2007c, 2018a). It also explores the construction of regular polygons and their applications in creating tessellations. The construction of regular polygons establishes a basis for comprehending symmetry and geometric relationships, which are subsequently utilized in more intricate fractal designs. Students investigate both static fractal constructions and those generated using dynamic geometry software. The primary components of the Unit 2 are outlined in the subsequent research procedures:

- What are tessellations, also known as tilings, of a plane? How can we create regular polygons and tessellations using regular polygons? Additionally, how can we construct regular polygons that are inscribed or circumscribed within a circle, particularly in relation to the number π (pi)?
- Digital methods for creating fractal structures, such as utilizing a dynamic geometry software environment like 'The Geometer's Sketchpad' or ‘Ultra Fractal’ (https://www.ultrafractal.com/).
- Iterative processes and associated calculations: - Perimeters and areas of fractal figures - Sequences, limits, the sum of the initial ‘n’ terms of a geometric progression and the infinite sums of geometric progressions.
- Tessellations and tilings utilizing fractals - Employing regular polygons and fractal designs for tiling.

Unit 3: Similarity and Self-Similarity

This unit represents the *Explicitation phase* of the FDP Program and expands upon the principles of geometric transformations, similarity and self-similarity, especially concerning fractals. Learners investigate geometric transformations such as rotation, reflection, and translation, and their significance in the creation of tessellations. By exploring self-similar tessellations, such as those found in *reptiles*, students gain insight into the iterative characteristics of fractals. Furthermore, the unit investigates the transformations of *pentominoes* and their relevance in the creation of tessellations. Calculations related to side lengths, areas, and *tangram puzzles* are explored, both in static and dynamic environments. The unit also introduces the construction of *Polyhedral nets and Platonic solids*, fostering spatial reasoning and deeper geometric understanding. The primary components of the Unit 3 are outlined in the subsequent research procedures:

- Geometric transformations, including rotation, reflection, and translation.
- Utilization of the coordinate plane to construct a two-dimensional geometric figure. Determination of the coordinates of these figures and their transformations as the figure is altered on the plane. The assessment worksheets will encompass the application of transformations in tessellations, employing both dynamic geometry software and conventional paper-and-pencil techniques.
- Transformations of pentominoes and tangram for creating tessellations. The construction and calculation of side lengths and areas of these figures utilizing both static and dynamic tools.
- Distinctions between central and axial symmetry as outlined in the official mathematics curriculum.
- Repetitive tessellations and self-similar tessellations, such as those found in reptiles.
- The construction of polyhedral nets and the theoretical exploration and geometric constructions of Platonic and Archimedean solids.

Unit 3 also includes iterative processes, allowing students to explore algebraic, trigonometric, and geometric concepts (e.g., through the use of Baravelle spiral or Dragon Curve).

Unit 4: Spirals

This unit represents the *Free orientation phase* of the FDP Program. Learners utilize their prior knowledge to solve problems, address challenges and explore more open-ended assignments. In this unit, students construct Baravelle spirals (Choppin, 1994) *linked with increasing or decreasing sequences plotted on plane* (e.g., Patsiomitou, 2005a, 2008g), as well as their related geometric properties, such as infinite series (Fig. 19). An examination of the Fibonacci sequence and the Golden ratio (φ) reveals their significance in both natural and mathematical spirals. The Golden Spiral and Fibonacci Spiral serve as prime illustrations of the relationship between fractals, self-similarity, and geometry. Learners also interact with Pascal's Triangle and algebraic identities, thereby enhancing their comprehension of mathematical patterns. The

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primary components of this unit are outlined in the subsequent research questions and procedures.

- What are spirals?
- Construction of a Baravelle spiral beginning the construction from either an equilateral triangle or a square. Calculations of side lengths, perimeters, areas, the sum of infinite terms of a geometric progression, the *n*th partial sum of a geometric sequence are a several parts of the instruction.
- The golden ratio (φ) – The Fibonacci sequence – Golden rectangles.
- The Golden Spiral, The Fibonacci Spiral.
- Pascal's Triangle – Algebraic identities.
- Golden rectangles and the golden spiral.
- Fibonacci sequence and Pascal's triangle.

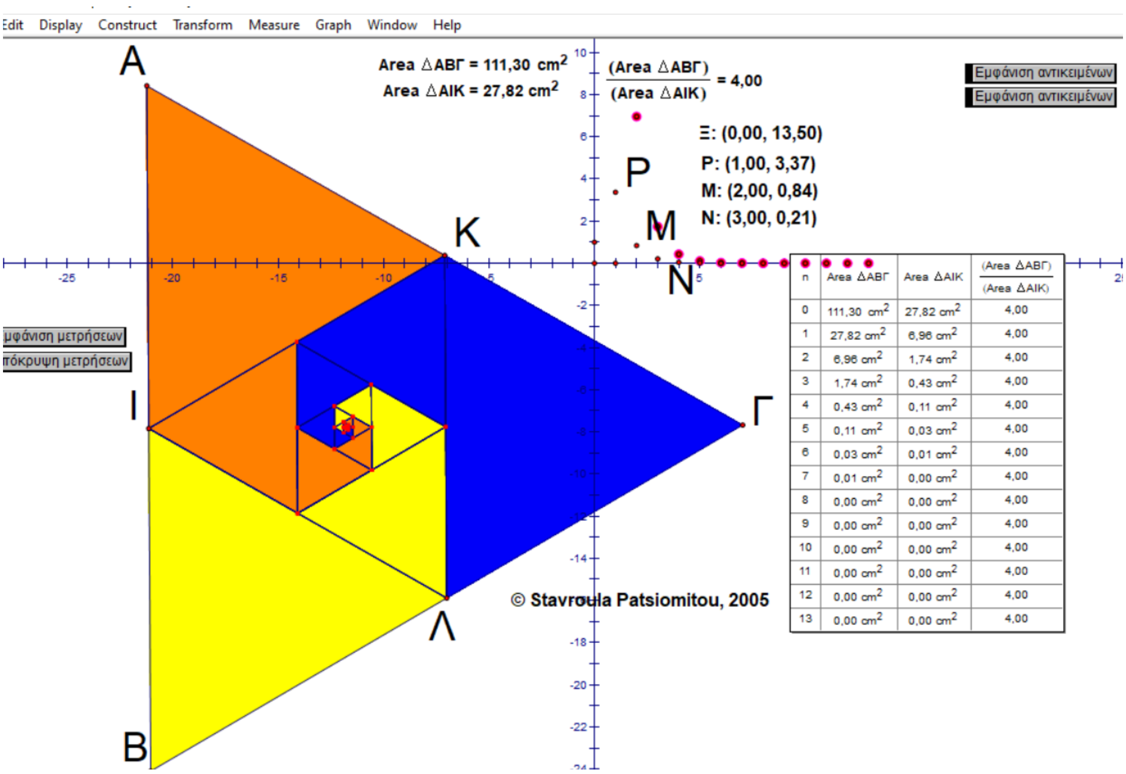

Figure 19: Constructing a Baravelle spiral that links the process to the concept of a decreasing geometric progression of successive areas (Patsiomitou, 2005a, pp.77-78)

I designed *instrumental trajectories* for the creation of increasing or decreasing sequences (see for example Patsiomitou, 2009b, 2021b). With regard to Baravelle spirals we can create an (e.g., Patsiomitou, 2005a, pp.77-78, 2007b, 2008g, 2009b, 2022a):

- **Instrumental learning path A**: we start from one side of the triangle and proceed with the process within the interior of the triangle. Subsequently, the series of measurements and calculations that arises is in a descending order.

- **Instrumental learning path B**: We bring parallel lines to the sides of the triangle resulting in the formation of larger triangles. Subsequently, the series of measurements and calculations that arises follows an ascending order.

This unit also emphasized the significance of generalization in facilitating algebraic or calculus reasoning.

Unit 5: The Pythagorean Tree Fractal-Modeling 3D fractal objects

This unit represents the *Integration phase* of the FDP Program. The Pythagorean Theorem and its converse serve as the foundation for this unit. Students engage with the construction of the Pythagorean Tree fractal, calculating side lengths, perimeters, and areas of successive triangles and squares. This unit provides a hands-on approach (i.e., a practical method) to understanding the Pythagorean Theorem and its applications in both static and dynamic geometric scenarios. By engaging in iterative fractal processes, learners uncover the significance of self-similarity within mathematical relationships. Students synthesize and integrate their knowledge, forming a new network of objects and relationships. The primary components of this unit are outlined in the subsequent research questions and procedures.

- The Pythagorean Theorem – The converse of the Pythagorean Theorem

- Construction of the Pythagorean Tree fractal. Calculations of side lengths, perimeters, and areas of successive triangles and squares, etc.

- Modeling 3D fractal objects-Polyhedral nets for Platonic and Archimedean solids.

- Engaging students in practical activities where they design kites using Baravelle spirals or Sierpinski triangles, as well as constructing three-dimensional models.

This final unit applies the concepts learned throughout the FDP Program to solve real-world problems involving fractal objects. Students create polyhedral nets for Platonic and Archimedean solids, utilizing these geometric forms to investigate real-world applications that foster creativity and practical problem-solving abilities (for instance, designing kites). The unit concludes with the construction of 3D models. For example, the process of modeling a giant Sierpinski triangle in plane evolves in different phases (Patsiomitou, 2013): Understanding the concept of fractal objects and exploring the Sierpinski triangle through visual representations. (b) Modeling the Sierpinski triangle on paper within a paper-and-pencil environment, and realizing the difficulty of progressing into the interior of the shape through iterative processes.(c) Constructing an equilateral triangle, identifying its midpoints, and creating a custom tool for the dynamic geometry software *Geometer's Sketchpad* (Patsiomitou, 2007).(d)

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Constructing a Sierpinski triangle in a dynamic environment by applying the custom tool [e.g., on an interactive whiteboard].(e) Constructing and repeating the process in the lab with physical materials such as cardboard, geometric tools, scissors, glue. (f) Repeating the structure of the initial Sierpinski triangle in order to accurately model the structure of the shape on a two-dimensional plane.

Figure 20*:* Collaborating with FDP groups for the modelling process, utilizing a synthesis of hand-on activities and digital means (Patsiomitou, 2012c, 2021d)

The students faced challenges not only in the construction of the tool and its application, but also, they encountered difficulty in implementing the appropriate method for the configuration of the final form of the Sierpinski triangle in the schoolyard (Fig. 20). This is attributed to the existence of a multiform modeling process. Specifically, (a) modeling based on the dynamic figure in a static manner, and (b) modeling derived from the mental image they had constructed in a dynamic environment by interpreting with natural materials. These various processes enabled students to create interconnected visual representations in their minds, facilitated by their engagement with the interactive visual representations provided by dynamic geometry software. This engagement ultimately enhanced their capacity for structural analysis of shapes and the translation of their mental images into tangible representations. Additionally, the interaction with semi-preconstructed (or preconstructed) *Linking Visual Active Representations* (e.g., Patsiomitou, 2007a, 2008a, b, c, d, e, f, g, h, 2009 a, b, c, d, e, f, g, h, 2010, 2011 a, b, 2012 a, b, c, d, 2013a, b, 2014, 2015a, b, c, 2016 a, b, c, 2018a, b, 2019a, b, c, 2020 a, b, c, 2021 a, b, 2022 a, b, c, d, 2023 a, b, c, 2024a, b, c, d) in the Geometer's Sketchpad software contributed to the development of both structural and conceptual abilities, enabling students to produce interconnected representations both mentally and procedurally. The idea of solving real-world problems has been highlighted by researchers (e.g., Burkhardt, 1981; Pierce & Stacey, 2009) as crucial for enhancing the understanding and learning of mathematical concepts. On the other hand, it is well-known that applying Mathematics to solve real-world problems is a complex process that involves a number of phases, as described by Corte, Verschaffel, & Greer (2000): understanding the situation described, constructing a mathematical model, working with the mathematical model, interpreting the results in the real-world context, and evaluating the outcome (p.71). To conclude: the activities and investigations related to transformational geometry and fractal geometry enabled the participating students to acquire an understanding of the essence of fractal geometry, while also engaging in significant generalization tasks that arose from the construction process. This innovative approach interconnects various fields of mathematics in an interdisciplinary manner, resulting in a rich, dynamic learning experience for students. Consequently, this FDP Program can be implemented in parallel with the official curriculum proposed by the Ministry of Education, as it combines mathematical theory, dynamic software tools, and practical applications to deepen students' comprehension of geometry, algebra, and calculus. The proposed FDP serves as a model for enhancing learning experiences in secondary, as well as higher education:

- Regarding *the geometric concepts* addressed in the current FDP, the following topics were discussed: Operations and calculations with line segments, comparisons and calculations of angles, concepts related to the circle, calculations of segment lengths and perimeters, equality of shapes (triangles, polygons), similarity of shapes, symmetry properties of figures, properties of basic quadrilaterals and the centroid of a triangle, hierarchy of quadrilaterals, Thales' theorem and its converse, the Pythagorean theorem and its converse, theorems of inscribed angles, theorems of regular polygons, the ratio of areas in relation to the ratio of similarity of sides, etc.
- Regarding *the algebraic concepts*, addressed in the current FDP, the following topics were discussed: Measurement scales, fractions, decimal numbers and the interrelationship among the three forms of rational number representations, ratios, proportions, properties of ratios [e.g. Invert Endo property a/b=c/d/ then b/a=d/c], computations involving algebraic expressions, sequences, geometric progressions, exponential functions, limits, sequences, limits of sequences, graphical representations of sequences, infinitesimals, the concept of infinity, etc.

11.0 COGNITIVE PSYCHOLOGY AND TEACHINC STRATEGIES

The Fractal-based Dynamic Program (FDP) presented here was built upon my direct teaching experiences with students. By integrating theoretical mathematics with practical, dynamic tools, this project provides a comprehensive and engaging way to learn about fractals, self-similarity, and their applications. For the FDP Program, as I mentioned, I developed instructional plans, activity sheets, and worksheets related to *Transformation Geometry*, intended for use in an interactive learning environment. The activities were meticulously tailored to align with the cognitive levels, developmental stages, and ages of the students, employing the van Hiele model (e.g., Fuys et al., 1984). Students develop a solid understanding of mathematical concepts in geometry, algebra, and pre-calculus, while also enhancing their critical thinking and problem-solving skills. They also engage in learning through enjoyable activities and interaction with mathematical tools. The application of fractals serves as an excellent medium for investigating the concepts of transformation geometry, including rotation, translation, iteration, and similarity. Additionally, it provides an engaging method to bridge abstract mathematical theories with visual and digital representations. Furthermore, what is the significance of *cognitive psychology* in relation to the instructions given during the FDP? Attention plays a critical role in how students perceive and identify figures, particularly in complex or abstract visual tasks. Without focused attention, especially in tasks involving sub-figures or empty [or negative] spaces, learners may fail to "see" what's actually present in the image. This aligns with theories such as *Feature Integration Theory* (Treisman & Gelade, 1980), an excerpt of their study is mentioned here:

> […] *features are registered early, automatically, and in parallel across the visual field, while objects are identified separately and only in a later stage, which requires focused attention. We assume that the visual scene is initially coded along a number of separable dimensions, such as color orientation, spatial frequency, brightness, direction of movement. In order to recombine these separate representations and to ensure the correct synthesis of features for each object in a complex display, stimulus locations are processed serially with focal attention*.

In the field of cognitive psychology, attention serves to filter incoming sensory information. In the context of geometric perception or figure recognition and discrimination— particularly in tasks involving the recognition and discrimination of figures such as polygons embedded within complex visual configurations —students *selectively attend*) on specific spatial characteristics while ignoring irrelevant or overlapping information and details. The *Feature Integration Theory* posits that attention is essential for combining individual visual elements, into a unified, coherent object. Complementarily, Gestalt principles (Claudia, 2009) propose that perceptual processing tends toward the organization of stimuli into complete and meaningful wholes. From an educational standpoint, this suggests that instructing students on how to direct their attention—such as through guided discovery or visual scaffolding—can significantly enhance their ability to perceive and recognize complex geometric structures. To assist students in identifying geometric figures, particularly when they are embedded in complex visual contexts, a teacher or an educator may implement a variety of strategies grounded in cognitive psychology and educational research.

In summary, the teacher's role is to systematically guide, assist, and progressively transfer control of students' visual focus, enabling them to analyze complex visual information. The objective is to empower students to not only observe but also to understand how to observe effectively. In other words, a teacher's goal is to help students not just see, *but to learn how to visualize 'hidden' information in figures.*

12.0 CONCLUSION

Looking ahead, the Fractal-based Dynamic Program enhanced with instrumental schemata has the potential to be further developed, refined and expanded/broadened to encompass more sophisticated topics related to fractals, including the exploration of the Mandelbrot set and fractal dimensions. Additionally, there is an opportunity to incorporate advanced technology, such as coding and simulation tools, enabling students to generate their own fractals using programming languages like Python or JavaScript. Broadening the Fractal-based Dynamic

Program to investigate the application of fractals in disciplines such as environmental science, engineering, and economics could offer students even greater chances to utilize their knowledge. In summary, the Fractal-based Dynamic Program represents an innovative and exciting approach to mathematics education, merging profound mathematical concepts with practical applications and interactive, experiential (dynamic, hands-on) learning. By engaging/ involving students in the study of fractals, the curriculum not only deepens their mathematical comprehension but also enhances/ cultivates critical thinking, creativity, and a sense of discovery (the spirit of reinvention). I completely concur with the interpretation conveyed in the subsequent text: *[…] fractals could meaningfully be integrated into the school curriculum to support students' conceptual understanding of infinite geometric series, rather than simply being a novel add-on to the established curriculum. It is hoped that teachers will find the integration of contemporary mathematics into the regular curriculum a valuable pedagogy in terms of nurturing students' curiosity and illustrating the evolving nature of the discipline*." (Shriki &Nutov, 2016, p.42)

Concerning the concept of 'instrumental schemata': both learners and instructors develop 'instrumental' schemata, indicating that these schemata are dynamic rather than fixed, undergoing continuous development within a digital or AI environment. (e.g., Patsiomitou, 2024b). Effective instruction in visual attention not only aids in perceptual tasks but also cultivates students' capacity for strategic visualization, a critical skill in mathematics education. Students have the opportunity to develop their thinking processes, a subject that will be examined in a forthcoming study.

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