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REIMAGINING SUSTAINABLE SCHOOL EDUCATION THROUGH A FRACTAL-BASED DYNAMIC PROGRAM (FDP): AN EXPERIENTIAL AND INTERACTIVE APPROACH

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ABSTRACT

The paper presents the design, development, and implementation of a Fractal-based Dynamic Program (FDP), which synthesizes findings and design principles derived from my prior empirical research with newly generated evidence. The FDP is organized around a learning trajectory comprising five instructional units, each briefly outlined in terms of its learning goals, activities, and conceptual progression. In the methodology section, I describe the essential elements of the FDP that I designed and implemented, and I also propose strategies for its effective classroom implementation. The study is situated within a design-based research methodology, adopting a qualitative case study perspective. The FDP is proposed as a flexible informal curriculum framework grounded in principles of transformation geometry and fractals, and designed to support interdisciplinary, sustainability-oriented school education. The implementation of the FDP demonstrates how fractals can function as a powerful interdisciplinary teaching context that connects mathematics and art while enhancing students' engagement and enjoyment of learning. Moreover, the study identifies pedagogical principles emerging from the implementation process, particularly concerning dynamic visualization, transformation-based reasoning, and students' gradual transition from empirical exploration to formal mathematical thinking. Finally, the study addresses the need to empower educators to design and mediate learning experiences that foster sustainability citizenship in school education.

Keywords: Fractal-Based Learning, Transformation geometry, Dynamic Geometry, Curriculum Design

1.0 INTRODUCTION

This study addresses the need to empower educators to design and mediate learning experiences that foster sustainability citizenship in school education. Sustainability citizenship is understood as active, participatory, and responsibility-driven engagement of students, educators, and school communities in sustainable practices (e.g., Tilbury, 1995; Hasrul, & Irawan, 2023). As noted by Hasrul, & Irawan (2023, p. 2), beyond its pedagogical dimensions, the integration of citizenship within sustainability education promotes experiential learning methodologies, encouraging students to engage in real-world sustainability projects and community initiatives.

The strong inclination of students to engage with digital technologies has prompted researchers and designers of educational settings to develop appropriate educational and pedagogical technological environments that facilitate and encourage innovative teaching and learning approaches for mathematical concepts. Within this context, educators must acquire the skills and knowledge necessary to enhance their effectiveness in teaching, as outlined by Mishra, & Koehler (2006). These authors identify several types of knowledge that teachers should possess when designing learning situations: technological knowledge, content knowledge, pedagogical knowledge, technological pedagogical knowledge, pedagogical content knowledge, technological content knowledge, and technological pedagogical content knowledge (TPACK).

> *The TPACK framework describes how effective teaching with technology is possible by pointing out the free and open interplay between technology, pedagogy, and content. Applying TPACK to the task of teaching with technology requires a context-bound understanding of technology, where technologies may be chosen and repurposed to fit the very specific pedagogical and content-related needs of diverse educational contexts* (Kereluik, Mishra, & Koehler, 2010; Mishra & Koehler, 2009 cited in Koehler et al., 2013).

In addition, emerging educational contexts increasingly require technological–pedagogical knowledge related to the use of Artificial Intelligence (Patsiomitou, 2024a, 2024b).
Dynamic Geometry Systems (DGS) are microworlds designed to facilitate the teaching and learning of Euclidean geometry, algebra, and calculus. Two-dimensional dynamic geometry software (DGS) packages include *Geometer's Sketchpad* (Jackiw, 1991/2001), *Cabri II* (Laborde, Baulac, & Bellemain, 1988), *GeoGebra* (Hohenwarter, 2001, 2002; Hohenwarter et al., 2008), and *Cinderella* (Richter-Gebert & Kortenkamp, 1999). Three-dimensional systems include *Cabri 3D* (Laborde, 2004), among others. In my study *"An 'Alive' DGS Tool for Students' Cognitive Development"* (Patsiomitou, 2018b), I report several important effects of DGS software on students' thinking: First, sequential linking pages can be created so that the entire dynamic file becomes an "alive book" (Patsiomitou, 2005a, p. 63; Patsiomitou, 2018b). These "alive digital representations" (Patsiomitou, 2005a, p. 67) enable the figural diagram to become dynamic, allowing students to focus their attention on *simultaneous modifications and transformations of objects on the screen* (e.g., Patsiomitou, 2005a, p. 68).

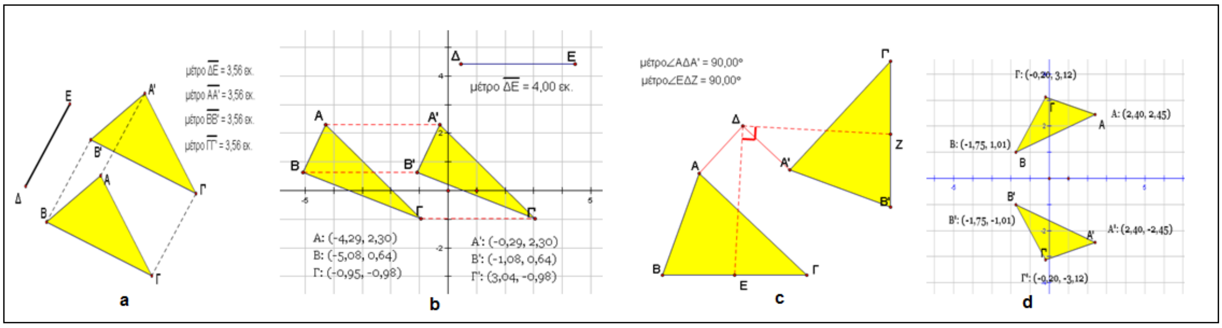

Figure 1: Transformations and use of coordinate planes in the DGS environment (Patsiomitou, 2009a, 2022)

The transformation of an object on screen using dragging can be combined with other techniques to cause a combination of transformations on screen (e.g., Patsiomitou, 2009a, b,

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2010, 2012a, b): (a) dragging and tracing objects (b) dragging and measuring objects (c) dragging and animating objects (d) dragging a transformed object or its image (through rotation, translation, or reflection). More complex combinations are also possible, such as dragging, tracing, and animation, or dragging, measuring, and rotating. These combinations of interaction techniques facilitate the visualization of mathematical concepts (e.g., Patsiomitou, 2008a, 2008b, 2010, 2011), as *a way we can "see" what is unseen* (Arcavi, 2003).

Moreover, this study promotes innovative instructional approaches that integrate environmental awareness while emphasizing experiential (e.g., Kolb, 1984; Kolb & Kolb, 2005), and learner-centered learning. It also explores innovative perspectives on learning through play. From the perspective of many researchers, play supports children's socio-emotional development, provides meaningful feedback, and stimulates cognitive growth through sensory experiences derived from the natural environment, as well as through the mental representations children construct and the ways these representations evolve over time. Play has also been closely associated with imagination and creativity (e.g., Lieberman, 1977; Piaget, 1951). In this context, the present study proposes approaches and activities grounded in play and in experiential, active, and socio-constructivist learning, with the aim of fostering a sustainable school environment.

In the methodology section, I delineate the essential elements of an innovative Fractal-based Dynamic Program (FDP) that I designed, developed, and implemented (e.g., Patsiomitou, 2007a, 2007b, 2016a, 2016b, 2016c, 2025). The project demonstrates how fractals can serve as a powerful interdisciplinary teaching tool that connects mathematics and art while enhancing students' enjoyment of learning. I also propose strategies for its effective implementation, as it has the potential to function as an informal curriculum centered on the principles of transformational geometry and fractals within educational projects. The aim of this study is to explore how fractal-based dynamic tasks can support students' mathematical meaning-making. The study adopts a design-based research approach (e.g., Brown, 1992; Cobb et al., 2003) drawing on classroom observations and analysis of students' work produced during the implementation of the Fractal-based Dynamic Program (FDP).

It is anticipated that educators will view the incorporation of fractal geometry into the standard curriculum as a valuable didactic and pedagogical framework for fostering students' curiosity and highlighting the dynamic nature of the field. The proposed activities are interdisciplinary, collaborative, and engaging, enabling students to assume meaningful roles and connect prior knowledge with new concepts. The integration of manipulatives and digital tools, together with guided mediation, supports the understanding of abstract concepts and strengthens the connection between theory and practice.

As mentioned above, the study proposes strategies for experiential, active, and socio-constructivist learning within a sustainable school framework. This involves "*developing competencies that empower individuals to reflect on their own actions, taking into account their current and future social, cultural, economic, and environmental impacts, from both a local and a global perspective*" (UNESCO, 2017, p. 7). At this point, the following research question is posed: *Which principles were employed during the implementation of the FDP, and how did they contribute to its successful completion?*

2.0 METHODOLOGY

2.1. The Fractal-based Dynamic Program (FDP)

The study is situated within a design-based research methodology (e.g., Brown, 1992; Cobb et al., 2003), which focuses on the iterative design, implementation, and refinement of educational interventions in authentic classroom settings. Design-based research integrates theoretical development with practical classroom experimentation, aiming to produce both effective learning environments and transferable design principles. In this study, the Fractal-based Dynamic Program (FDP) was designed, implemented, and analysed through iterative teaching cycles of classroom application, allowing for continuous reflection and adaptation of the *learning trajectory* (e.g., Simon, 1995; Battista, 2011; Patsiomitou, 2012b). Simon (1995) developed the idea of a *teaching cycle* and created a diagram in order to represent the way that a [hypothetical] learning trajectory is an ongoing modification of three components: "(a) the learning goal that defines the direction, (b) the learning activities and (c) the hypothetical learning process" (p. 136). In addition, the design-based research approach was complemented by a qualitative case study perspective, enabling an in-depth analysis of students' interactions, critical events, and meaning-making processes within the dynamic geometry environment.

This section introduces the innovative *Fractal-based Dynamic Program (FDP), Fractals – From Zero to Infinity* (initially presented/published in Greek in Patsiomitou, 2016a, 2016b), which can function as an informal curriculum centered on the concept of fractals. The program aims to integrate fractal geometry into secondary education through the use of dynamic mathematical tools. It encompasses a range of mathematical domains, including geometry, algebra, and pre-calculus, explored through experiential learning activities (e.g., Kolb, 1984; Kolb & Kolb, 2005), interactive digital resources, and problem-solving tasks. I submitted the FDP proposal for approval to the [Greek] Governing Committee of Model and Experimental Schools (D.E.P.P.S.) for the school years 2011–2012, 2012–2013, and 2013–2014 (Patsiomitou, 2025). The program consisted of two hours of instruction per week, scheduled after school. The participants were students from various school-level classes, aged between 13 and 15 years. The description of the Fractal-based Dynamic Program (FDP) represents a synthesis of both an *instructional design and an instructional redesign process* (Patsiomitou, 2012b, p. 58). Besides the design and redesign of the FDP, I also served as the instructor of this program for three consecutive years. Instructional design is not conceived as a linear or static process; rather, it is approached as an ongoing cycle of design, implementation, and redesign. Within this cycle, I continuously adapted the learning trajectory in response to emerging classroom needs.

To structure instruction, I designed and followed a sequence of five flexible phases: (a) *Phase 1:* Recalling and activation of pre-existing knowledge through a brief review of key concepts using targeted questions (5 minutes), (b) *Phase 2:* Exploration and experimentation with active student involvement/participation, utilizing dynamic geometry software or hands-on manipulatives tools (10 minutes), (c) *Phase 3:* Formulation and mathematization, where empirical observations are transformed into mathematical expressions and logical arguments (10 minutes), (d) *Phase 4:* Application and feedback through short problem-solving tasks (10 minutes), and (e) *Phase 5:* Reflection and closure, aimed at enhancing students' metacognitive skills (10 minutes). This framework functioned as a flexible guide rather than a rigid instructional sequence. Instruction took place in a classroom equipped with an interactive whiteboard, where students worked collaboratively in small groups, often sharing a single computer interface.

Gagné, Briggs, and Wager (1992) proposed a systematic instructional design framework, *The Events of Instruction and their Relation to Processes of Learning*, grounded in a behaviorist perspective. Although my own stance is constructivist, I consider the principle of “gaining attention” to be relevant across instructional contexts. In classroom practice, capturing students’ attention—particularly those who perceive geometry as difficult or unappealing—is essential. By engaging them in the construction of aesthetically meaningful (“beautiful”) mathematics, I was able to “gain” their attention and support subsequent processes of meaning-making.

I sought to engage students’ attention by introducing mathematically rich tasks that supported the construction of meaning. Through the FDP program, *Fractals – From Zero to Infinity*, I provided students with opportunities to experiment, investigate, and deepen their understanding of mathematical concepts encountered both in formal schooling and through digital environments. Over the three-year implementation, I used fractal structures as a bridge between mathematical reasoning, artistic expression, and game-based learning, with the primary objective of fostering students’ conceptual understanding of geometric ideas through experiential, collaborative, and creative activities.

The study was further informed by the work of Dina van Hiele-Geldof (1957/1984) and van Hiele (1986), who examined how instructional methods influence learning outcomes. Whereas Piaget (1975/1985) emphasized biological maturation in cognitive development, van Hiele highlighted the role of structured instructional phases in the development of geometric thinking. These phases include: *Information (Inquiry), Directed Orientation, Explicitation, Free Orientation, and Integration* (Fuys et al., 1984, p. 251).

The instructional design was grounded in inquiry-based learning (e.g., Jaworski, 2003) and incorporated *Linking Visual Active Representations* (e.g., Patsiomitou, 2008a, b, 2010, 2019), alongside static representations and manipulative models. I engaged students in constructing fractal objects using both paper-and-pencil methods and digital tools, enabling them to recognize the limitations of static media and conventional geometric instruments in representing fractal structures. Students frequently compared traditional geometric tools with digital software environments, identifying the affordances and limitations of each. Vygotsky (e.g., 1934/1962) emphasizes the significant role of social knowledge construction and the influence of language on student's ability to articulate their thoughts. Scaffolding is a concept that Vygotsky introduced and elaborated upon, viewing the use of tools as essential supports for enhancing students' cognitive processes. The progression of language as students advance through various levels is also a characteristic of the van Hiele model. This model addresses the disruptions and cognitive conflicts that arise as students evolve their thinking across different stages. How can students enhance their thinking according to the van Hiele framework? By utilizing manipulatives during the initial phases of their development and adhering to an instructional sequence designed to scaffold their language skills. As students refine their thinking, they also improve their language usage and formulation: initially, they predominantly rely on *inductive reasoning*; however, as they progress and engage in a structured instructional approach, they begin to employ *abductive and deductive reasoning* (e.g., Peirce, 1992; Simon, 1996; Patsiomitou, 2019).

Drawing on my experience as a mathematics teacher, I designed instructional materials that anticipated students' thinking, or sought to "imagine a route by which [students] could have arrived (or could arrive) at a personal solution" (Gravemeijer & Terwel, 2000, p. 780). This approach is consistent with the principle of reinvention (Freudenthal, 1973), as well as with work in Dynamic Geometry System (DGS) environments guided by the principle of dynamic reinvention (Patsiomitou, 2012b, 2014). In my studies, I emphasize the impact *of Linking Visual Active Representation (LVAR) modes* (e.g., Patsiomitou, 2008a, 2008b, 2010) on students' gradual development of competence in constructing rigorous mathematical proofs during problem-solving processes. I argue that the dynamic reinvention of knowledge refers to the type of knowledge that students reconstruct through interaction with LVAR artefacts created within a DGS environment—knowledge "for which they themselves are responsible" (Gravemeijer & Terwel, ibid.).

Furthermore, the interdisciplinary approach has proven to be more effective than the traditional method of instruction (e.g., Wang, 2012). For instance, as noted by Patsiomitou (2016c, p. 985), in the following scenario the cultivation of broccoli Romanesco can inspire students to take on roles that align with various specialties: Group C (students acting as Chemists) assesses soil acidity; Group B (students serving as Biologists) examines which plants are appropriate for the specific soil; Group M (students assuming the roles of Mathematicians) monitors plant growth over time to create charts and analyze development; Group R (students operating as Researchers) investigates environmental factors in relation to the plant's growth in both international and local contexts; Group S (students acting as Historians/Sociologists) studies the plant's history, associated myths, and ties to local traditions; Group P (students working as Photographers) methodically documents plant growth through photography; and lastly, Group A (students portraying Actors) presents a performance related to the theme (for example, comedic stories that illustrate the process).

I developed the instructional units of the FDP over a six-month period, using the Geometer's Sketchpad DGS environment. The materials draw on chapters from my book *Learning Mathematics with Geometer's Sketchpad v4* (Patsiomitou, 2009a, 2009b), which was updated in 2022 under the title *Conceptual and Instrumental Trajectories Using Linking Visual Active Representations Created with the Geometer's Sketchpad* (Patsiomitou, 2022). These monographs synthesize and extend my earlier research presented at conferences in Greece and published in academic journals. Although minor adaptations occurred during implementation, the learning outcomes I observed among students were consistently positive. In particular, the constructions and explorations conducted within the Geometer's Sketchpad dynamic geometry environment (DGS) form the basis of the instructional units presented in the following section. Overall, the FDP Units section provides a concise overview of the informal curriculum that I designed and implemented.

2.2. The FDP Units

Unit 1: The Road to Infinity – From Zeno of Elea to Benoit Mandelbrot (*An Introductory Historical Overview*).

This unit corresponds to the *Information (Inquiry) phase* of the FDP Program and provides a historical introduction to the development of fractal ideas, from ancient Greek philosophy to modern mathematics. Through the study of classical mathematical fractals—such as the Sierpinski Triangle, Sierpinski Carpet, Menger Sponge, and Koch Snowflake—students develop an understanding of fractal structures and their infinite properties. The unit also introduces key mathematicians including Benoit Mandelbrot, Georg Cantor, and Konstantinos Karatheodory, as well as the concept of fractal (fractional) dimension (e.g., Löfstedt, 2008) and its implications for Euclidean geometry.

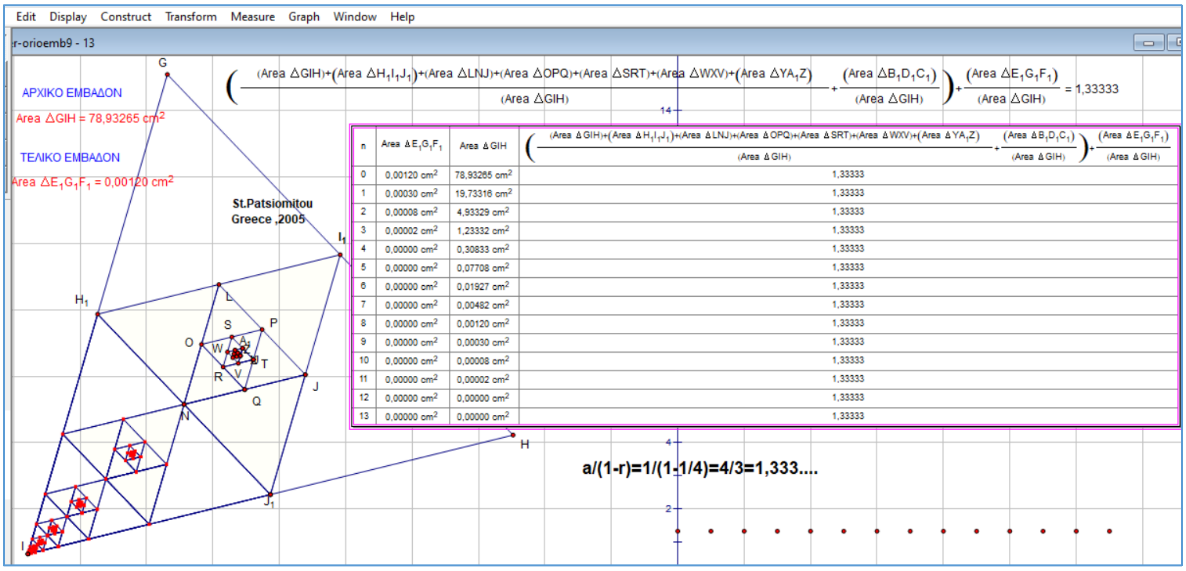

Figure 2: Generalization of meanings by using the iteration process (Patsiomitou, 2005a, pp.72-74)

Students examine examples of natural and mathematical fractals and construct fractal figures on the plane using both traditional geometric tools and dynamic geometry environments. Students also investigate iterative processes, including calculations involving perimeters and areas of fractal figures, sequences and limits, and sums of geometric progressions. The modeling and construction of an in-depth fractal structure is difficult or impossible with familiar geometry instruments (ruler and compass). Although students initially began their constructions in a paper-and-pencil environment, they found it necessary to continue their work using dynamic geometry software. The construction of the Sierpinski triangle fractal emerged as one of their most engaging activities. To construct the Sierpinski triangle, students started with an isosceles (or equilateral) triangle and identified the midpoints of its sides (Patsiomitou, 2005a, 2005b, 2006a, 2006c, 2007a, 2007b). I then guided them in developing a custom tool to automate and extend the iterative process. Through this approach, I introduced pre-calculus concepts by building on students' existing mathematical knowledge. In particular, iterative processes and the associated calculations of perimeters and areas of fractal figures were used to introduce the concepts of sequences, limits, the sum of infinite terms of a geometric progression, and infinite geometric series.

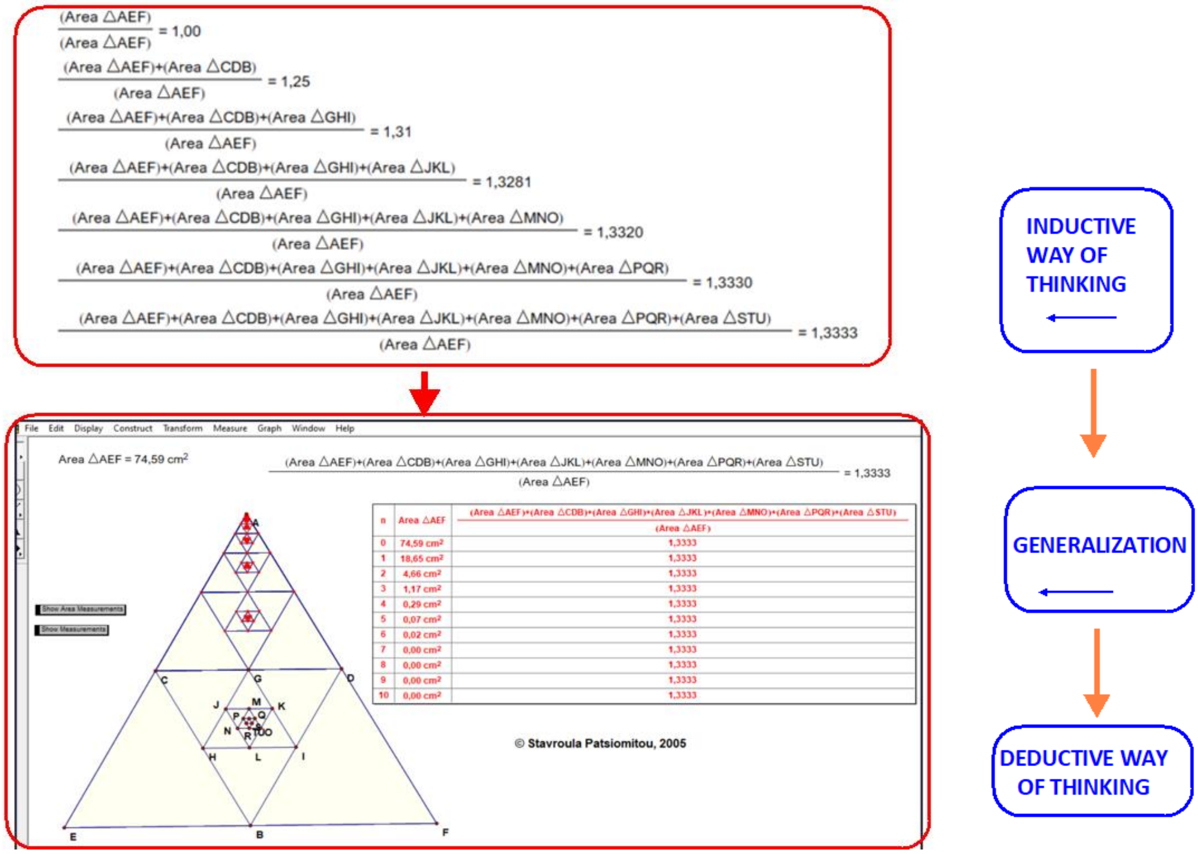

Figure 3: Inductive thinking, Generalizations and Deductive thinking (Patsiomitou, 2005a, pp.72-74)

For example, I posed the following task: *Construct an equilateral triangle with area E= 1 cm² and calculate/determine its side length. Create the midpoints of its sides and calculate/determine the area* of the resulting inner equilateral triangle. *What do you observe happening to the sum of the terms in the sequence of areas when they are divided by the area of the original triangle?* To what value does this sum tend? Does it converge to 1.33333…, and if so, why?

Particular emphasis is placed on the observation of what are termed *critical events* (e.g., Rotem, & Ayalon, 2022; Tirosh, Tsamir, Levenson, & Barkai, 2019), defined as unpredictable moments in teaching that reveal how students interpret mathematical knowledge. In the words of Rotem, & Ayalon (2022): *Critical events are moments in which students' mathematical thinking becomes apparent and thus can provide teachers opportunities to delve more deeply into the mathematics discussed in the lesson.* Such events serve as a starting point for teacher reflection and are closely linked to the concept of *didactic transposition* (Chevallard, 1989), namely, the transformation of scientific knowledge into forms that are pedagogically accessible and meaningful for students. For example, the same result can be obtained by applying the procedure with a different orientation. As observed, while students are initially guided to draw conclusions through inductive reasoning, they are subsequently led toward generalizations that form the basis for deductive thinking.

Within this framework, flexibility, pedagogical improvisation, and the capacity to adapt or replace tools and strategies in response to instructional or *instrumental obstacles* (e.g., Patsiomitou, 2011, 2019) are recognized as essential components of teaching competence. The primary components of this unit are articulated through the following research questions: (a) What are fractals? (b) Can mathematical fractals be constructed on the plane using traditional or dynamic geometry tools? What critical events emerge during the iterative process? (c) To what extent can the use of software (e.g., zooming tools) support students in spontaneously expanding their *concept images* (Tall & Vinner, 1981) to incorporate infinitesimal structures?

Unit 2: Tessellations of the Plane

This unit represents the *Directed orientation phase* of the FDP Program, as per the phases outlined in van Hiele's theory. Students engage in carefully designed exploratory activities such as folding, measuring, and constructing geometric figures. This phase facilitates the exploration of particular concepts, with tessellations, or plane tilings, being a fundamental focus in the study of geometry. The next activity was the transformation of a geometrical figure that was used as a building unit -a generator- for the construction and a means by which the students could construct the meanings of theorems inductively and experimentally, which were included in their class curriculum. For example, the rotation of a right triangle leads to the construction of a rectangle, then to the construction of a trapezoid, and, finally, to the construction of a right triangle whose sides are double from the sides of the initial triangle. This unit also introduces the concept of regular polygons, inscribed and circumscribed within a circle. Students explore the properties of these polygons and their relationship with the number pi (e.g., Patsiomitou, 2006b, 2018a).

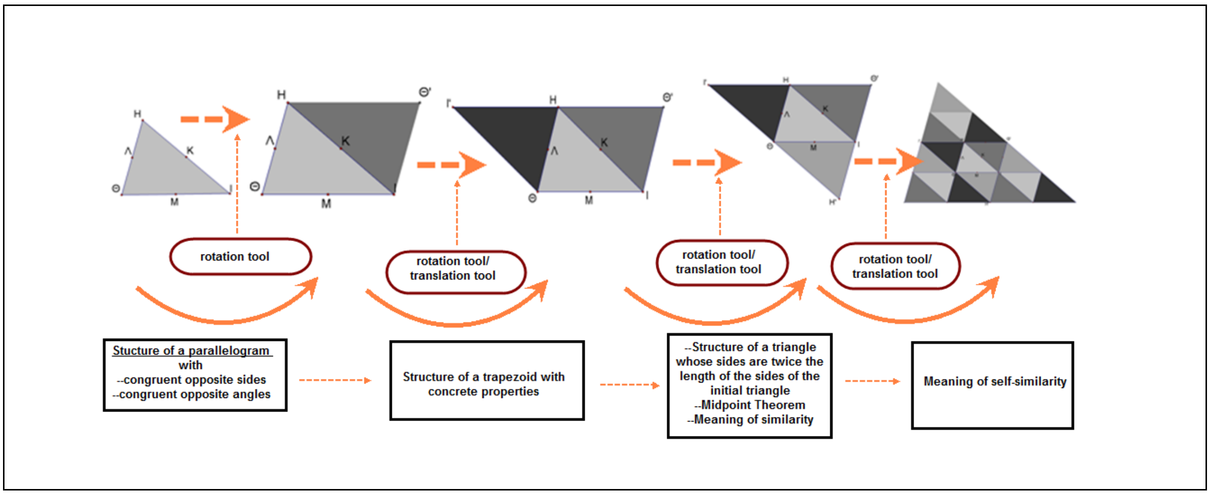

Figure 4: A triangle generator for the conception of the meaning of self-similarity (Patsiomitou, 2009b, as cited in Patsiomitou, 2014, p. 21)

It also explores the construction of regular polygons and their applications in creating tessellations. The construction of regular polygons establishes a basis for comprehending symmetry and geometric relationships, which are subsequently utilized in more intricate fractal designs. Following Felix Klein’s Erlangen Program (1872/1893), geometry is understood as the study of invariants under transformation groups. The transformation geometry of microworlds is grounded in Felix Klein’s definition of geometry, whereby the properties of geometric objects are invariant under groups of transformations. The use of software-based transformations led me to conclude (Patsiomitou, 2019) that "the method of superposition can be *instrumentally decoded* in terms of translation transformations, whereby a figure is transferred to another position in space through the use of a dynamic vector" (p. 11).

The most important part of this unit is the introduction of number pi in experimental way. Many students—especially those with low achievement in mathematics—struggled to distinguish between the image of a 60-gon and that of a circle, as both appeared visually similar in their textbook. The representation of the 60-gon created a cognitive conflict for most students. As this misunderstanding prevented further progression, I decided to modify my instructional approach and introduce the concept through a narrative based on how Archimedes (287–212 BC) might have reasoned. Archimedes *method of exhaustion* constitutes one of the earliest theoretical approaches to approximating the value of π. Specific examples from my experimental research using dynamic active representations have been analyzed in the methodology section of my study *A dynamic active learning trajectory for the construction of number pi (π): transforming mathematics education* (Patsiomitou, 2018a): (a) the construction of number pi as an approximation process. For this, I created in the Geometer's Sketchpad software the process of an inscribed or circumscribed n-gon in a circle with a view to using the tabularized measurements and calculations of a ratio in combination with the software's iteration process to lead the students to visualize the approximation process of number pi (Patsiomitou, 2006b); (b) the construction of number pi through Riemann sums in a DGS environment (Patsiomitou, 2006c); (c) the construction of number pi by means of a real world problem (Patsiomitou, 2013, 2016a, 2016b). For this, I combined a digital visit to the Guggenheim Museum in New York using the Google Earth software with dynamic representations of the Geometer's Sketchpad software and other digital web resources. My aim was the students to conceive the meaning of number pi as a limit using the iteration process of the Geometer's Sketchpad dynamic geometry software. Finally, the role the active representations play in the learning trajectory made me think of a way to define what a '*dynamic active learning trajectory*' is (Patsiomitou, 2018a).

In the Figure 5, a snapshot from the GSP file Curved.path.gsp [5] is presented, which I have redesigned, enriched with elements resembling a real-world setting, such as fractal trees, a blue sky, and a green hill on which a wheel-based activity was situated. The wheel begins its motion at the starting point and stops at the top of the hill. By counting the number of rotations of the wheel, students can calculate the distance between the two points. This activity proved to be highly engaging and enjoyable for students across age groups. The formulation of engaging and original problems within a dynamic geometry environment captures students' interest and encourages them to develop ideas and techniques through which they construct mathematical concepts and "discover" solutions. Learning to construct mathematical concepts is a demanding process that requires substantially more practice than the mere acquisition of accumulated information.

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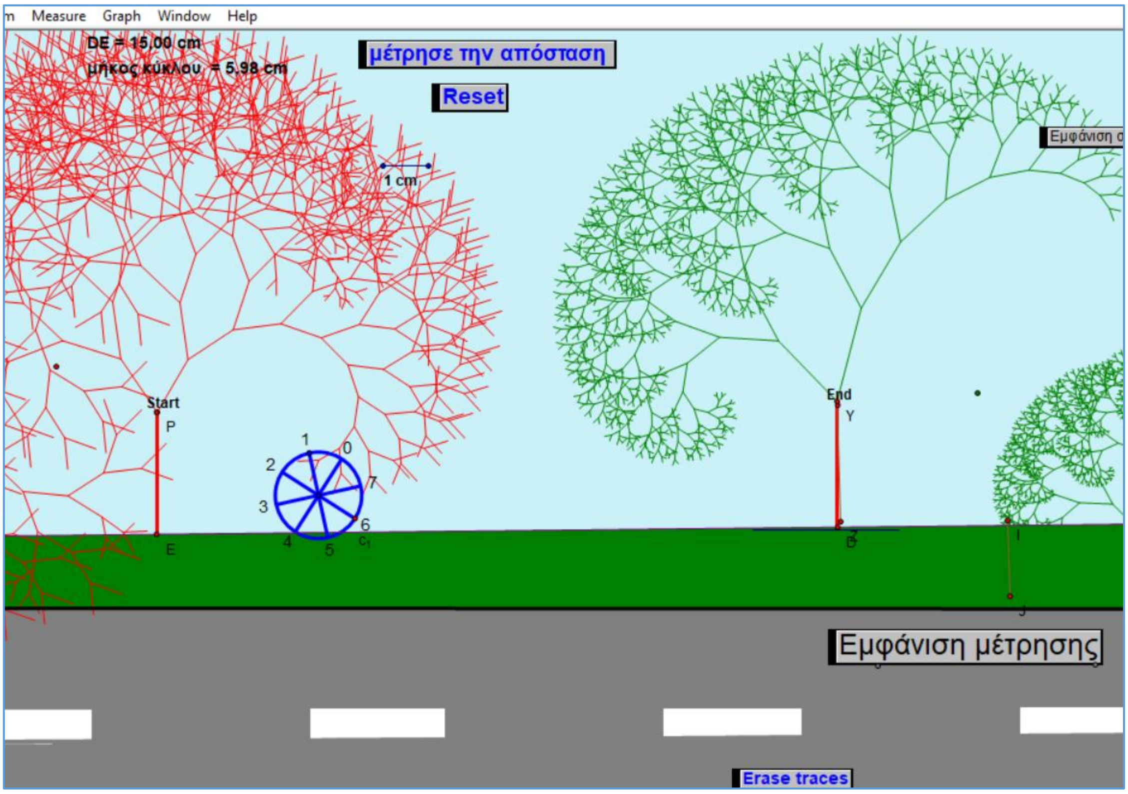

Figure 5: Calculating the distance between two points along a curved path (Patsiomitou, 2012c, p. 146)

The primary components of this unit are outlined in the subsequent research questions: (a) What are tessellations, also known as tilings, of a plane? (b) How can we create regular polygons and tessellations using regular polygons? (c) How can we construct regular polygons that are inscribed or circumscribed within a circle? Can we construct tessellations utilizing fractal objects?

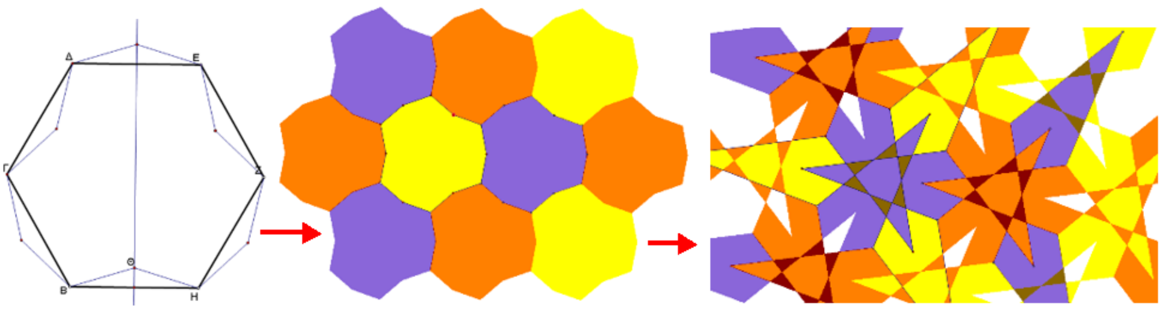

Figure 6: The process of creating a tessellation through successive rotations in a playful way (Patsiomitou, 2009c)

Unit 3: Similarity and Self-Similarity

This unit corresponds to the *Explicitation phase of the FDP Program* and extends students' understanding of geometric transformations, similarity, and self-similarity. Students explore

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transformations such as rotation, reflection, and translation, and apply them in the construction of similarity and self-similarity objects. Using both paper-and-pencil techniques and dynamic geometry software, learners investigate the coordinates of geometric figures and analyze how these coordinates change under transformations.

This exploration is connected to the work of Benoit Mandelbrot (1975) and supported by visualizations in dynamic software environments, which provide students with rich opportunities to observe self-similarity within mathematical objects. Furthermore, the unit investigates the transformations of *pentominoes* and their relevance in the creation of tessellations. Calculations related to side lengths, areas, and *tangram puzzles* are explored, both in static and dynamic environments. The study is further extended through the presentation of images from nature, whose structures are characterized by self-similarity (e.g., broccoli, fern). Iterative processes are further investigated through fractal structures such as the Baravelle spiral (Choppin, 1994) and the Dragon curve. Additional activities include the construction of polyhedral nets and the study of Platonic and Archimedean solids, supporting the development of students' spatial reasoning. At this stage, students are able to recognize patterns and properties in natural forms that they were previously unable to observe or reproduce. By exploring self-similar tessellations, such as those found in *reptiles* (Golomb, 1964) students gain insight into the iterative characteristics of these constructions (e.g., Patsiomitou, 2009a, 2009b, 2009c, 2009d).

Transformations on prototype elements (e.g., points, line segments) lead the students to (1) visualize the objects that are constructed in the first phase of the process and (2) perceive a few properties of the figure's symmetry initially at the visual level. It is observed that students connect, in their minds, representations that help them to respond to the next level, according to the theory of van Hiele. Therefore, *dynamic geometric transformations* are defined (Patsiomitou, 2014, p. 31): *as the rearrangement or modification of the diagram on screen that result in the modification in one or more embedded geometric objects. This could be an elicitation from the addition, cancelation of the diagram's elements that cause the rearrangement of the diagram, its anasynthesis, its metamorphosis or* even from the alteration of an object's size or orientation. In the words of Dina van Hiele (1984) the diagram goes through a *metamorphosis* as a result of the manipulations of reconfigurations "followed by a phenomenological analysis and an explicating of its properties: it becomes what we call a [dynamic] geometric symbol" (Dina van Hiele in Fuys et al., 1984, p.221). Moreover, *a metamorphosis could be seen as we apply one or more interaction techniques, or their combination, on the diagram's objects.* The difficulty students experience in imagining transformations of geometric figures during problem-solving situations is rooted in the nature of geometrical concepts, which Fischbein (1993) defined as an amalgam of "abstract ideas on one hand and sensory representations reflecting some concrete operations on the other" (p. 14).

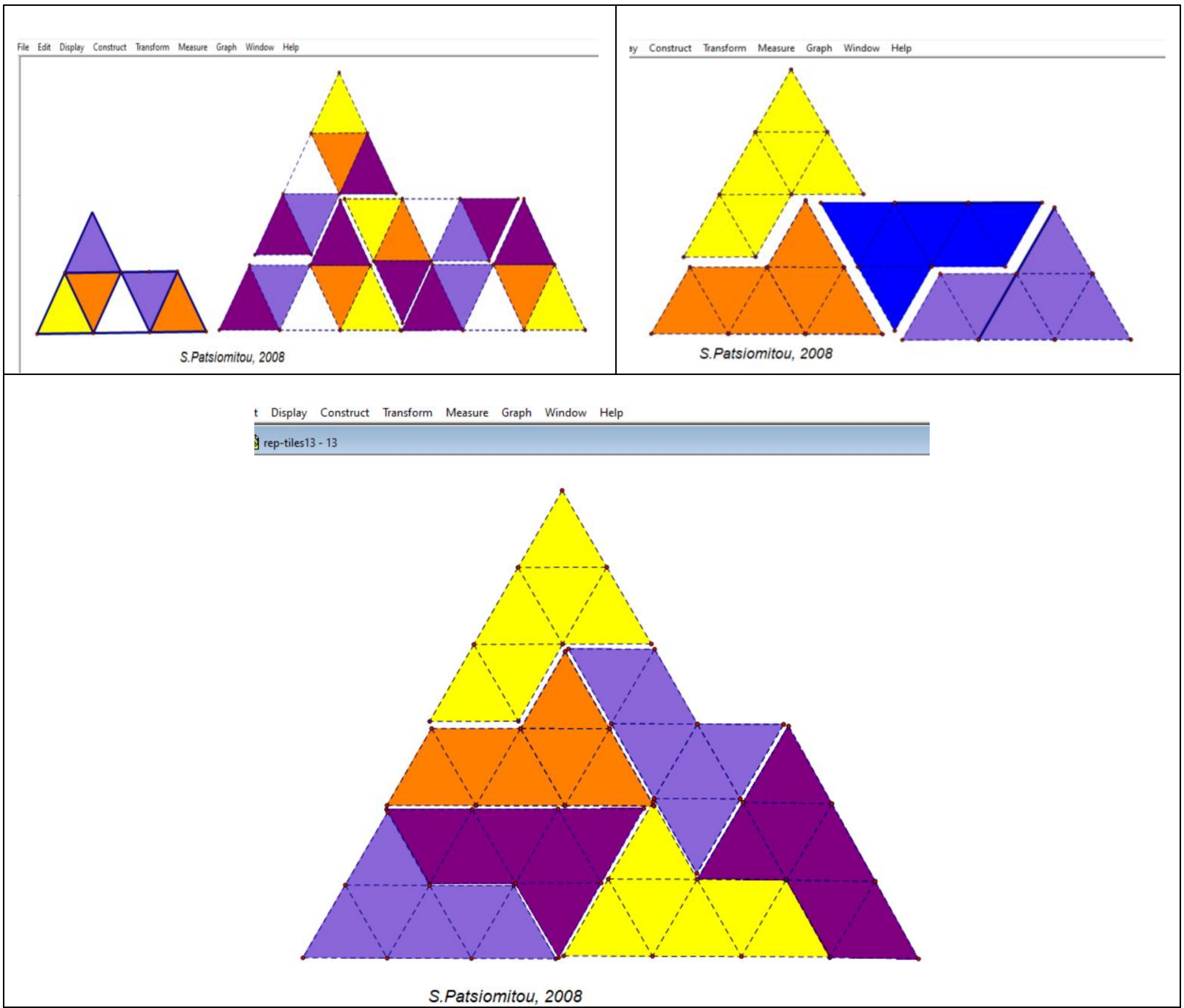

Figure 7 a, b, c: Tessellations using rep-tiles

The primary components of the first part of this unit are outlined in the subsequent research questions: (a) Can you determine the coordinates of a two-dimensional geometric figure constructed in the coordinate plane and their transformations as the figure is altered on the plane by rotation or translation etc. (employing both dynamic geometry software and conventional paper-and-pencil techniques)? (b) Can you use transformations of pentominoes, tangram, reptiles for creating tessellations? (c) Can you calculate the areas of these figures? (d) Can you find self-similarity shapes in nature?

Unit 4: Spirals

This unit corresponds to the *Free Orientation phase of the FDP Program*, where students apply their previously acquired knowledge to explore more open-ended mathematical problems. In this unit, students construct Baravelle spirals *linked with increasing or decreasing sequences plotted on plane* (e.g., Patsiomitou, 2005a, 2008d, 2009b, 2019, 2025), and investigate their associated geometric properties, including infinite series. I designed *instrumental trajectories* (e.g., Patsiomitou, 2019, 2021a, 2023b) for the creation of increasing or decreasing sequences.

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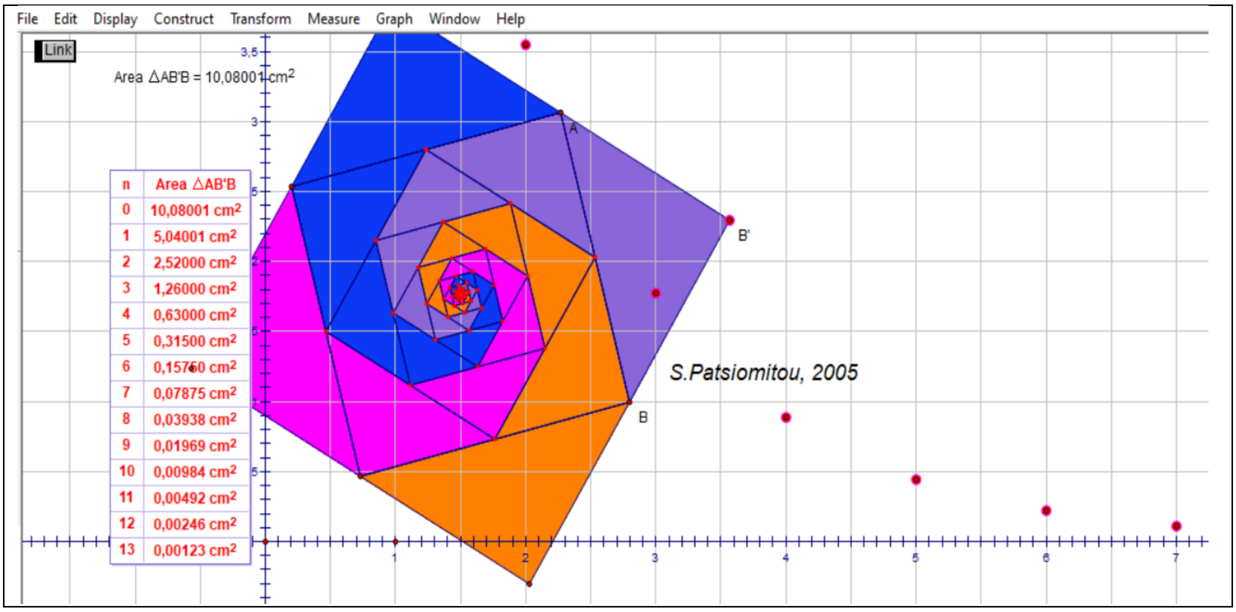
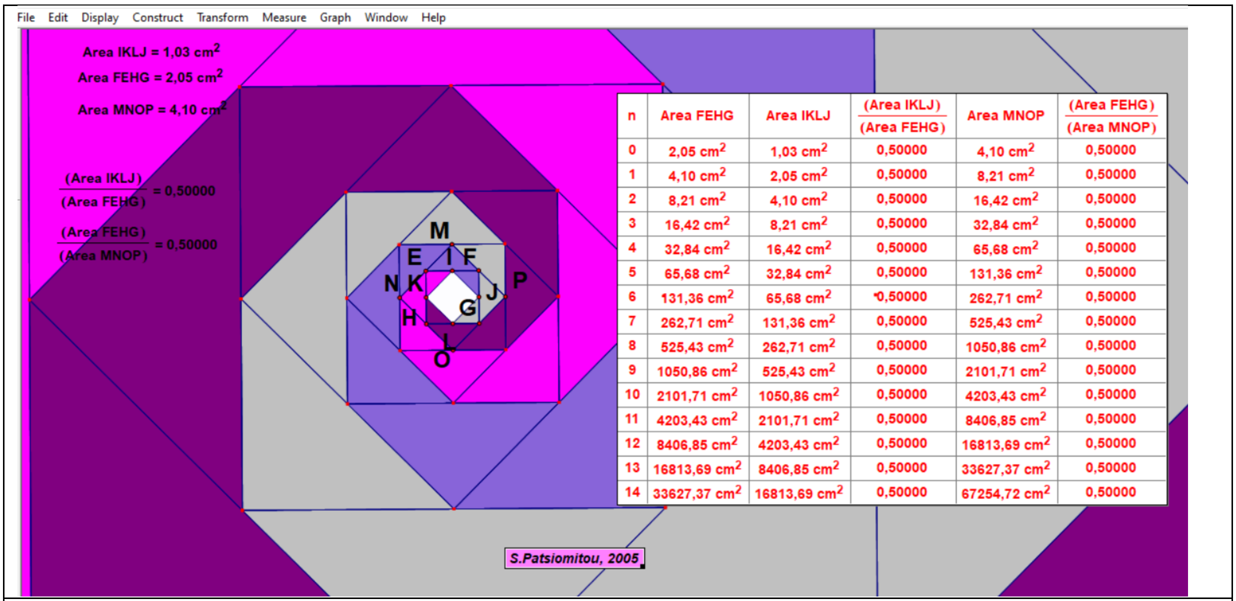
For example, concerning triangular Baravelle spirals, we are able to generate (e.g., Patsiomitou, 2005a, pp.77-78, 2008d, 2009b, 2019, 2022, 2023b): (a) an *Instrumental learning path A:* if we start from one side of the triangle and proceed with the process within the interior of the triangle, the series of measurements and calculations that arises is in a descending order. (b) an *Instrumental learning path B:* If we bring parallel lines to the sides of the triangle, the series of measurements and calculations that arises follows an ascending order. This unit also emphasized the significance of generalization in facilitating algebraic or pre-calculus reasoning.

Moreover, incorporating fractals into the current curriculum enables students to enhance their creativity and utilize mathematics in practical, real-world situations where other students may not recognize its significance. For instance, among the kites created by students, there was a particularly notable one designed with Baravelle spirals (e.g., Patsiomitou, 2009b) fractals, made by a 12-year-old student who was part of the FDP group. The key aspect in this case is that (a) the student verbally and visually describes the construction process—a process I have termed *instrumental decoding* (e.g., Patsiomitou, 2011)—and (b) applies it in order to construct a kite. Specifically, the student developed design skills, understood as the translation of a mentally constructed image into a geometric representation, using geometric instruments or digital tools to produce a figure with specified properties. Furthermore, the student developed application skills in order to construct a mathematical model-game. At this point, we can state that the student has developed cognitive and kinesthetic /psychomotor control over the processes, in accordance with Bloom's taxonomy (Bloom et al., 1956). Additionally, students have enhanced their imagination as well as their geometric thinking—dimensions that are addressed within other theoretical frameworks (e.g., the van Hiele theory).

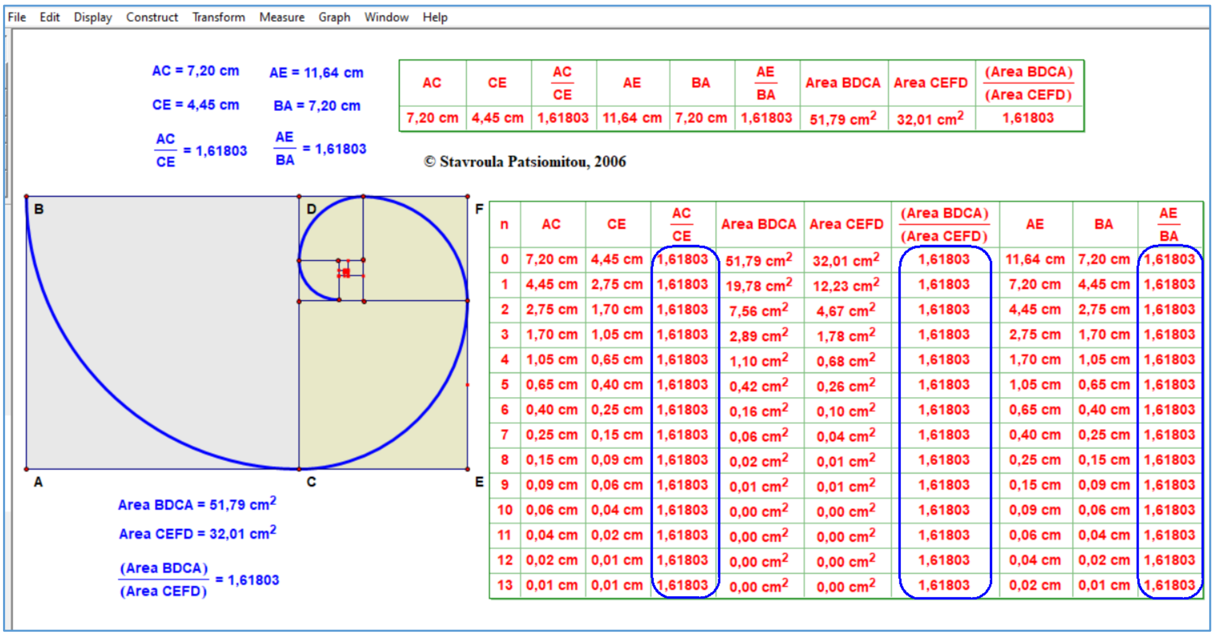




n	Area FEHG	Area IKLJ	(Area IKLJ) / (Area FEHG)	Area MNOP	(Area FEHG) / (Area MNOP)
0	2,05 cm²	1,03 cm²	0,50000	4,10 cm²	0,50000
1	4,10 cm²	2,05 cm²	0,50000	8,21 cm²	0,50000
2	8,21 cm²	4,10 cm²	0,50000	16,42 cm²	0,50000
3	16,42 cm²	8,21 cm²	0,50000	32,84 cm²	0,50000
4	32,84 cm²	16,42 cm²	0,50000	65,68 cm²	0,50000
5	65,68 cm²	32,84 cm²	0,50000	131,36 cm²	0,50000
6	131,36 cm²	65,68 cm²	0,50000	262,71 cm²	0,50000
7	262,71 cm²	131,36 cm²	0,50000	525,43 cm²	0,50000
8	525,43 cm²	262,71 cm²	0,50000	1050,86 cm²	0,50000
9	1050,86 cm²	525,43 cm²	0,50000	2101,71 cm²	0,50000
10	2101,71 cm²	1050,86 cm²	0,50000	4203,43 cm²	0,50000
11	4203,43 cm²	2101,71 cm²	0,50000	8406,85 cm²	0,50000
12	8406,85 cm²	4203,43 cm²	0,50000	16813,69 cm²	0,50000
13	16813,69 cm²	8406,85 cm²	0,50000	33627,37 cm²	0,50000
14	33627,37 cm²	16813,69 cm²	0,50000	67254,72 cm²	0,50000

Figure 8a, b: Linking representations in a Baravelle spiral construction (Patsiomitou, 2008d, 2009b, 2019, 2025)

As I report in previous studies (e.g., Patsiomitou, 2007a, 2009b, 2014, 2018b, 2019, 2022) the result of the process of iteration is the construction of the tables that repeat the process of initial measurements and calculations in dynamic connection with the shape, thus increasing (or decreasing) the level of the process of iteration while the software adds (or removes) the next level of measurements (or even calculations), whereas in the first column of the table the sequence of the natural numbers is presented.



AC	CE	AC / CE	AE	BA	AE / BA	Area BDCA	Area CEFD	(Area BDCA) / (Area CEFD)
7,20 cm	4,45 cm	1,61803	11,64 cm	7,20 cm	1,61803	51,79 cm²	32,01 cm²	1,61803

n	AC	CE	AC / CE	Area BDCA	Area CEFD	(Area BDCA) / (Area CEFD)	AE	BA	AE / BA
0	7,20 cm	4,45 cm	1,61803	51,79 cm²	32,01 cm²	1,61803	11,64 cm	7,20 cm	1,61803
1	4,45 cm	2,75 cm	1,61803	19,78 cm²	12,23 cm²	1,61803	7,20 cm	4,45 cm	1,61803
2	2,75 cm	1,70 cm	1,61803	7,56 cm²	4,67 cm²	1,61803	4,45 cm	2,75 cm	1,61803
3	1,70 cm	1,05 cm	1,61803	2,89 cm²	1,78 cm²	1,61803	2,75 cm	1,70 cm	1,61803
4	1,05 cm	0,65 cm	1,61803	1,10 cm²	0,68 cm²	1,61803	1,70 cm	1,05 cm	1,61803
5	0,65 cm	0,40 cm	1,61803	0,42 cm²	0,26 cm²	1,61803	1,05 cm	0,65 cm	1,61803
6	0,40 cm	0,25 cm	1,61803	0,16 cm²	0,10 cm²	1,61803	0,65 cm	0,40 cm	1,61803
7	0,25 cm	0,15 cm	1,61803	0,06 cm²	0,04 cm²	1,61803	0,40 cm	0,25 cm	1,61803
8	0,15 cm	0,09 cm	1,61803	0,02 cm²	0,01 cm²	1,61803	0,25 cm	0,15 cm	1,61803
9	0,09 cm	0,06 cm	1,61803	0,01 cm²	0,01 cm²	1,61803	0,15 cm	0,09 cm	1,61803
10	0,06 cm	0,04 cm	1,61803	0,00 cm²	0,00 cm²	1,61803	0,09 cm	0,06 cm	1,61803
11	0,04 cm	0,02 cm	1,61803	0,00 cm²	0,00 cm²	1,61803	0,06 cm	0,04 cm	1,61803
12	0,02 cm	0,01 cm	1,61803	0,00 cm²	0,00 cm²	1,61803	0,04 cm	0,02 cm	1,61803
13	0,01 cm	0,01 cm	1,61803	0,00 cm²	0,00 cm²	1,61803	0,02 cm	0,01 cm	1,61803

Figure 9: Measurements, calculations and graphical representation of the terms of the sequence regarding Golden ratio (φ) (Patsiomitou, 2006c, 2008c, 2023b)

In that way through this operation, the environment of the software promotes the investigation of the sequences. The iteration process by functioning thus has integrated or embodied the meaning of sequence while there is a direct connection between the user's perception and the abstract mathematical meaning. The diagrams' reconfiguration through the complex synthesis of combinations of transformations can lead to a continuous interaction of *discursive, visual and operational apprehension* (e.g., Patsiomitou, 2011, 2012a, 2012b, 2013, 2014, 2018b).

Further exploration of the Fibonacci sequence and the Golden ratio (φ) highlight their significance in both natural and mathematical spirals (e.g., Patsiomitou, 2006c, 2008c, 2019, 2023a, 2023b). The Fibonacci spiral and the Golden spiral provide characteristic examples of the relationship between fractals, self-similarity, and geometric reasoning. Students also engage with Pascal's Triangle and algebraic identities, enhancing their understanding of mathematical patterns and structures.

In the following task, I sought to make aspects of the physical world more accessible to students. Using a Geometer's Sketchpad (GSP) file, I created a simulation of a leaf (Figure 10a), and a dynamic flower (Figure 10b) constructed as the combination of a circular arc and its reflection (Patsiomitou, 2023a). By manipulating parameters, I was able to scale and reproduce the structure dynamically. Additionally, I enabled the movement of individual leaves, allowing them either to separate or to approach one another. Such dynamic representations support students in developing experiential understandings of mathematical concepts and the associated cognitive schemata.

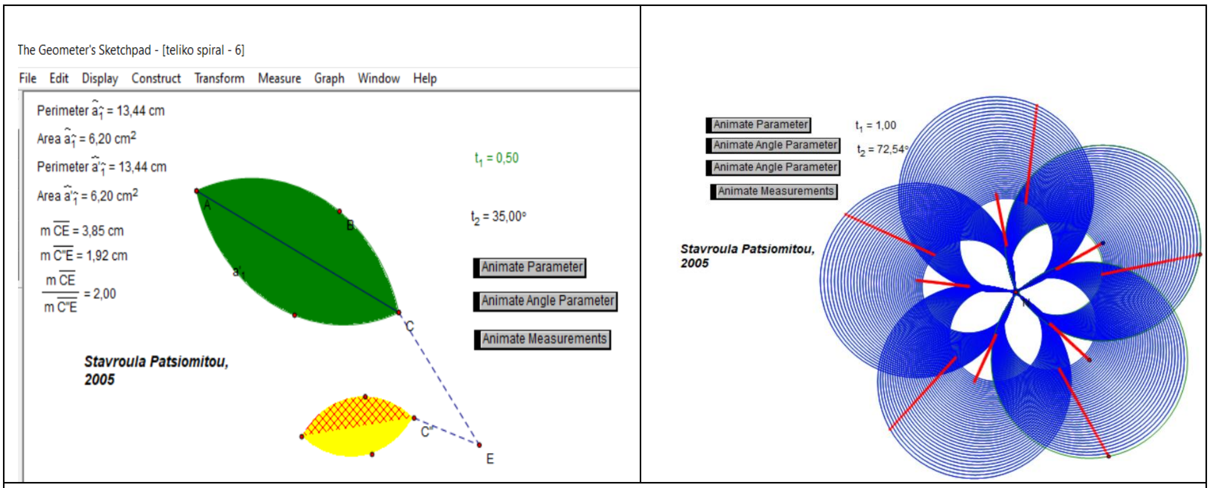


Figure 10a, b: A Dynamic flower and dynamic leaves (Patsiomitou, 2005a, 2022b, 2023a)

For instance, students may notice that a leaf is composed of two circular arcs, that it displays an axis of symmetry, and that the angles created between branches differ in various configurations. This task also enables students to explore angle transformations and variations in angle magnitude. The construction of a digital flower emerges from the generation of a spiral through systematic variation of parameters (Patsiomitou, 2005a, 2006a, 2023a).

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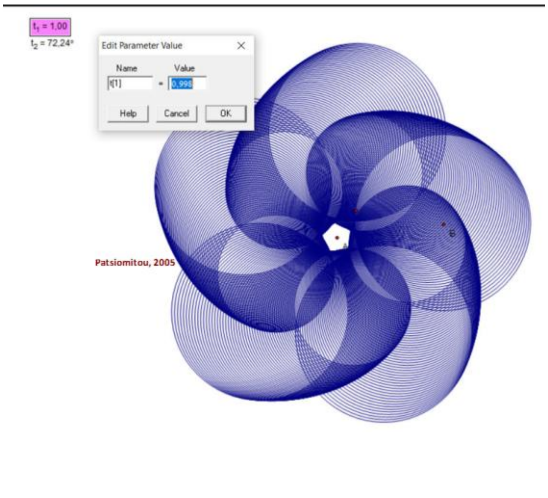

Figure 11: Construction of a spiral "dynamic flower" (Patsiomitou, 2005a, 2005b).

Figure 12: Spiral galaxy [1]

The same digital environment can be adapted for use with younger learners, supporting the observation and initial understanding of fundamental concepts such as angle, arc, and circle. This unit corresponds to the Free *Orientation phase of the FDP Program*. The primary components of this unit are outlined in the subsequent research questions and procedures. (a) What are spirals? (b) Can you describe the construction of a Baravelle spiral beginning the construction from either an equilateral triangle or a square? (c) Can you calculate the areas of the subsequent figures? (d) Describe The golden ratio (φ), The Fibonacci sequence and the Golden rectangles (e) Can you create an ascending or a descending sequence of terms?

Unit 5: The Pythagorean Tree Fractal-Modeling 3D fractal objects

This unit corresponds to the *Integration phase of the FDP Program*. Building upon the Pythagorean Theorem and its converse, students construct the *Pythagorean Tree fractal* and calculate side lengths, perimeters, and areas of successive geometric figures generated through iterative processes. Students synthesize the concepts developed throughout the program while exploring self-similarity in geometric relationships. Students synthesize and integrate their knowledge, forming a new network of objects and relationships.

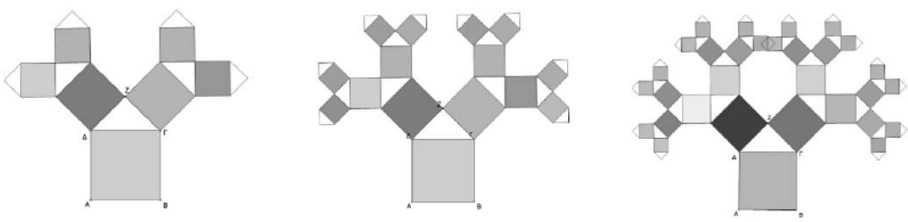

Figure 13: Sequential steps for the construction of a Pythagorean tree fractal

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Figure 13 is incorporated in my book (Patsiomitou, 2009b, 2022a), where I detail the sequential *instrumental trajectory* (Patsiomitou, 2021a, b), I used for the construction, using *linking visual active representations*. My primary goal was to investigate the ways in which the software assists students in creating patterns associated with the repetition of objects during the construction process, the formulation of measurements and calculations, and the positioning or reorientation of shapes on the plane. Furthermore, I examined the *generalization of the process*, considering the dynamic relationship between the fractal construction and the measurements/calculations in a linked table, along with its connection to graphical representations (Figure 14). Ghosh (2016) asserts that "*One of the foundational aspects of developing algebraic thinking is the ability to generalize. Research describes two kinds of generalization (Kinach, 2014), namely, generalization by analogy and generalization by extension. Generalization by analogy refers to observing a pattern, extending a sequence to the next few terms and being able to relate a particular term of the sequence to its previous terms. This kind of generalization requires recursive thinking. Generalization by extension, on the other hand, refers to writing a formula for the nth term of a sequence – which requires explicit thinking". (p.59).*

Hands-on activities encourage students to apply fractal concepts in creative contexts (such as designing kites based on Sierpinski triangles or Pythagorean trees). Attention plays a critical role in how students perceive and identify figures, particularly in complex or abstract visual tasks. Without focused attention, especially in tasks involving sub-figures or empty [/ negative] spaces, learners may fail to "see" what's actually present in the image (Patsiomitou, 2025).

This aligns with theories such as *Feature Integration Theory* (Treisman & Gelade, 1980): *We assume that the visual scene is initially coded along a number of separable dimensions, such as color orientation, spatial frequency, brightness, direction of movement. In order to recombine these separate representations and to ensure the correct synthesis of features for each object in a complex display, stimulus locations are processed serially with focal attention*. From an educational standpoint, this suggests that instructing students on how to direct their attention—such as through guided discovery or visual scaffolding—can significantly enhance their ability to perceive and recognize complex geometric structures.

To assist students in identifying geometric figures, particularly when they are embedded in complex visual contexts, a teacher or an educator may implement a variety of strategies grounded in cognitive psychology and educational research. The unit also includes the modeling of three-dimensional fractal objects through the construction of polyhedral nets for Platonic and Archimedean solids.

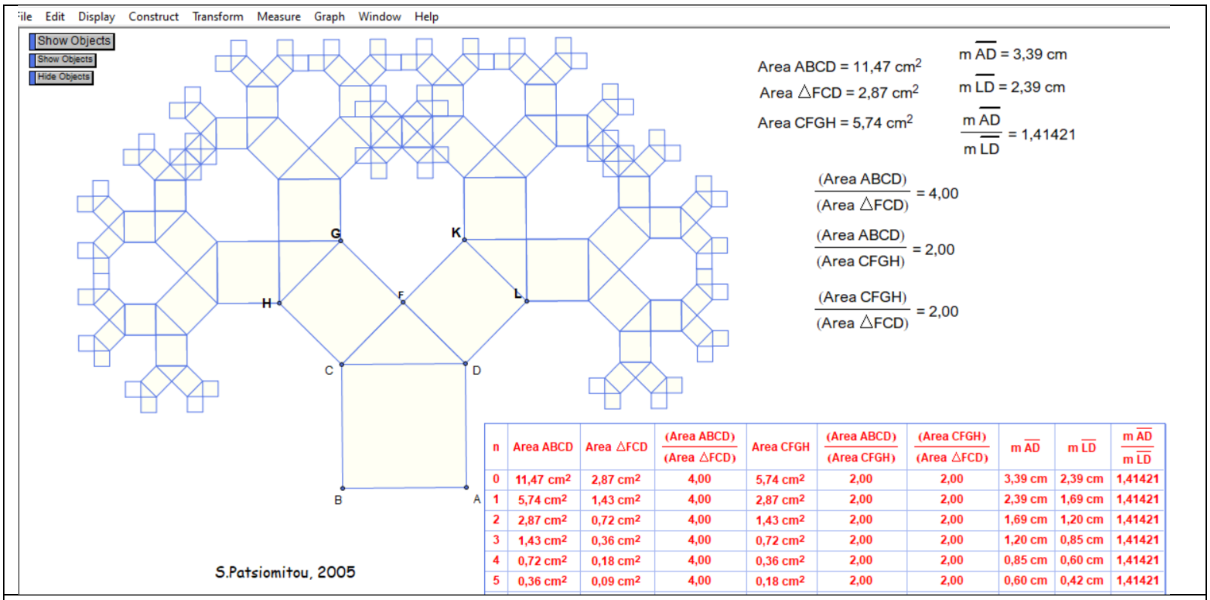

Figure 14: The Pythagorean tree structure dynamically linked with the table (Patsiomitou, 2007b, 2025)

In the illustration above (Figure 14), the figural representation of the fractal Pythagorean tree is dynamically linked to the measurements and calculations presented in the table. This means that when any point in the Pythagorean tree is manipulated, the corresponding measurements and calculations in the table are automatically updated. This dynamic relationship is important for the development of students' thinking, as it connects the visual representation with the symbolic representation of the same mathematical object. At this point we can pose questions to students such as: What is the relationship between the lengths of the sides of successive squares?

Table 1		
Iteration (n)	The length of a side of square Sn	The area of a square E_n
0 (initial)	$S_0=a$	$E_0= a^2$
1	$S_1=a :\sqrt{2}$	$E_1= E_0:2$
2	$S_2=a :2$	$E_2= E_1:2$
3	$S_3=a : 2\sqrt{2}$	…
4	$S_4= a :4$	
…	…	…
n	$S_n= a : (\sqrt{2})^n$	$E_n= E_{n-1}:2= a^2:2^n$

The lengths of these sides form a geometric progression, decreased by a constant ratio, the ratio of one of the congruent sides of the isosceles right triangle to the hypotenuse (since the legs of the isosceles right triangle are in ratio of $1:\sqrt{2}$ to the hypotenuse). Therefore, each successive square has a side length equal to $1:\sqrt{2}$ times that of the previous one. Throughout the iterative construction process, it becomes evident that all the resulting triangles possess both isosceles and right-angled properties. This iterative pattern reveals a consistent mathematical relationship among the triangles, squares, and other geometrical elements involved. In particular, when we calculate the lengths of the sides of these shapes, we observe that they decrease according to a geometric sequence, with each subsequent side being reduced by a factor of $1{:}\sqrt{2}$ in relation to its previous one. So, if the first square's side is equal to a then the next square's sides are: a, $a{:}\sqrt{2}$, a:2, a: $(2\sqrt{2})$… (Patsiomitou, 2025, p. 358).

In my study *A proposal for a fractal-based "Dynamic" Program: The Pythagorean Tree structure generated through instrumental schemata* (Patsiomitou, 2025) I extended my investigations for the fractal Pythagorean tree. In the illustration below, we observe the repetition of empty spaces [/negative spaces] shaped like irregular heptagons and cardioids, which follow a certain regularity that warrants further examination). As the figure expands, these units continuously decrease in size while increasing in number. This is a subject that requires further investigation: How does their number increase? What type of sequence regulates their development? The orientation of figures can significantly influence students' identification of shapes. Throughout the process, there is an increasing capacity and tendency to consider and analyze the spatial configuration of shapes by examining their individual components and the relationships between those components, along with an improved, enhanced ability to comprehend and apply formal geometric principles in analyzing and evaluating the interconnections and the interrelations of figures' properties.

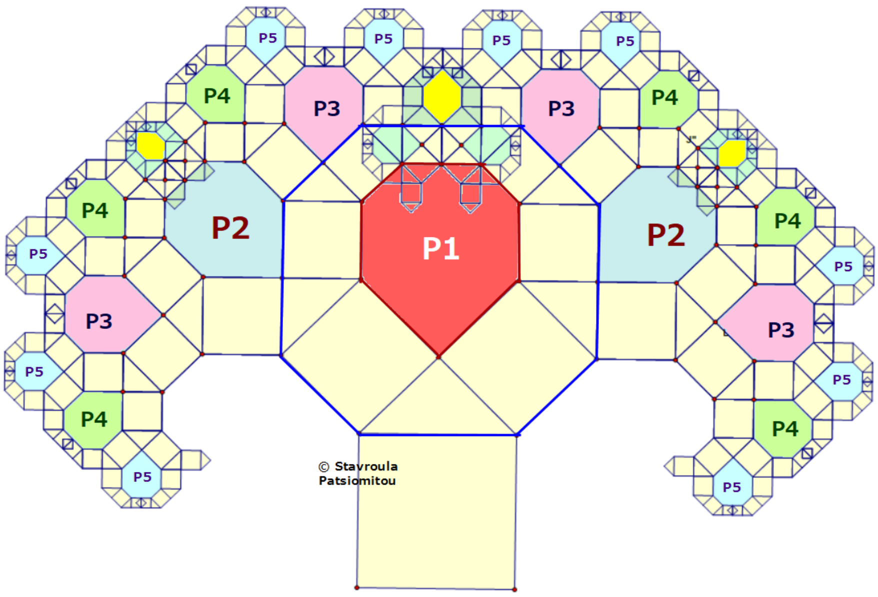

Figure 15. Irregular heptagons and octagons formed within the empty spaces between the branches (Patsiomitou, 2025, p. 361)

In the field of cognitive psychology, attention serves to filter incoming sensory information. In the context of geometric perception or figure recognition and discrimination— particularly in tasks involving the recognition and discrimination of figures such as polygons embedded within complex visual configurations —students *selectively attend*) on specific spatial characteristics while ignoring irrelevant or overlapping information and details. The *Feature Integration Theory* posits that attention is essential for combining individual visual elements,

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into a unified, coherent object. Complementarily, Gestalt principles (Claudia, 2009) propose that perceptual processing tends toward the organization of stimuli into complete and meaningful wholes. From an educational standpoint, this suggests that instructing students on how to direct their attention—such as through guided discovery or visual scaffolding—can significantly enhance their ability to perceive and recognize complex geometric structures. To assist students in identifying geometric figures, particularly when they are embedded in complex visual contexts, a teacher or an educator may implement a variety of strategies grounded in cognitive psychology and educational research.

3.0 RESULTS

The program concludes with the construction of physical and digital models (e.g., the step-by-step modeling of a large Sierpinski triangle, progressing from conceptual understanding and paper-based exploration to dynamic geometry construction and large-scale physical modeling using materials such as cardboard and geometric tools). Numerous student projects are included in the portfolio titled "*Didactic Approaches to Teaching Mathematics to Students with Different Learning Styles: Mathematics in the Real World*" (Patsiomitou, 2012c), as well as in the book "*Creativity and Skills in Mathematics*" (Patsiomitou, 2012b).

This is attributed to the existence of a multiform modeling process. Specifically, (a) modeling based on the dynamic figure in a static manner, and (b) modeling derived from the mental image they had constructed in a dynamic environment by interpreting with natural materials. These various processes enabled students to create interconnected visual representations in their minds, facilitated by their engagement with the interactive visual representations provided by dynamic geometry software. This engagement ultimately enhanced their capacity for structural analysis of shapes and the translation of their mental images into tangible representations. Additionally, the interaction with semi-preconstructed (or preconstructed) *Linking Visual Active Representations* in the Geometer's Sketchpad software contributed to the development of both structural and conceptual abilities, enabling students to produce interconnected representations both mentally and procedurally.

The proposed FDP serves as a model for enhancing learning experiences in secondary, as well as higher education (Patsiomitou, 2009a, 2009b, 2022, 2025): (a) Regarding *the geometric concepts* addressed in the current FDP, the following topics were discussed: Operations and calculations with line segments, comparisons and calculations of angles, concepts related to the circle, calculations of segment lengths and perimeters, equality of shapes (triangles, polygons), similarity of shapes, symmetry properties of figures, properties of basic quadrilaterals and the centroid of a triangle, hierarchy of quadrilaterals, Thales' theorem and its converse, the Pythagorean theorem and its converse, theorems of inscribed angles, theorems of regular polygons, the ratio of areas in relation to the ratio of similarity of sides, etc. (b) Regarding *the algebraic concepts*, addressed in the current FDP, the following topics were discussed: Measurement scales, fractions, decimal numbers and the interrelationship among the three forms of rational number representations, ratios, proportions, properties of ratios [e.g. Invert Endo property a/b=c/d/ then b/a=d/c], computations involving algebraic expressions, sequences, geometric progressions, exponential functions, limits, sequences, limits of sequences, graphical representations of sequences, infinitesimals, the concept of infinity, etc.

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Consequently, this FDP Program can be implemented in parallel with the official curriculum proposed by the Ministry of Education, as it combines mathematical theory, dynamic software tools, and practical applications to deepen students' comprehension of geometry, algebra, precalculus and calculus meanings.

4.0 DISCUSSION

The implementation of the FDP program highlighted several principles for designing effective learning environments that integrate mathematical inquiry, play, and interdisciplinary learning (e.g., Apostel, 1972). Involving educators and learners to the educational project FDP serves as an excellent model of *interdisciplinary collaboration*. The following principles have been introduced in a previous research study (Patsiomitou, 2016c) and they are briefly presented here:

A. It is essential to emphasize that activities should be aligned with the biological development and abilities of students at different ages, as well as with the skills they are expected to acquire when collaborating with peers and supervising adults, or when working independently. The aim is to ensure that students are not mentally overwhelmed. Games should also be adapted to the child’s developmental stage, following Piaget’s framework, since age plays a significant role in the development of emotional, kinaesthetic, and cognitive abilities. By gradually increasing conceptual complexity while respecting developmental readiness, educators can create learning environments where students actively construct understanding rather than passively receive information. Consequently, effective instructional design requires that learning activities be *systematically aligned with students’ developmental stages and cognitive capacities*, ensuring that play-based and inquiry-based experiences support meaningful conceptual growth. Consequently, this principle can therefore be summarized as: **Aligning activities with students’ developmental maturity.**

B. Children independently, or with gentle encouragement, choose the materials they use for play. They often select the space in which they will play, decide with whom they will play, and determine the progression of their play activities. It is important to allow children the opportunity to think and create a tangible outcome within a defined timeframe. Each child develops skills in different ways, as learning is promoted through discovery, experience, and pattern formation. By engaging in these self-directed experiences, learners develop critical thinking and creative problem-solving skills. Activities that encourage students to *construct conceptual maps* (Novak, & Cañas, 2006) of their reasoning further strengthen understanding and reflection, linking hands-on experience with conceptual knowledge. Consequently, this principle can be summarized as: **Promotion of children’s agency and engagement through self-selection, enabling autonomy and personalized learning.**

C. Children can be guided—or seek guidance—as often as they wish from a supervising specialist or educator, who facilitates their engagement through appropriate interventions, scaffolded questions, and prompts of both cognitive and creative nature (for example, through brainstorming sessions, problem solving and similar activities) (e.g., Schmidt, 1983; Burkhardt, 1988; Paulus, & Brown, 2007). Furthermore, it is important for supervising educators to intervene in order to assist a child in participating within a group, particularly when, for various reasons, the child has not assumed any role and becomes isolated, or when it is necessary to stimulate motivation (Driscoll & Nagel, 2002). Activities are intentionally designed to foster reflection, dialogue, and critical thinking, allowing students to translate hands-on experience into conceptual understanding. Consequently, this principle can be summarized as: **Collaboration and timely Intervention by supervisors as facilitators and designers of learning.**

D. Participation in activities *about, for, and within the environment* begins with engagement in environmental actions, and active involvement in group work. By participating in well-designed scenarios—such as environmental projects, simulations, or collaborative problem-solving tasks—students can explore concepts in real-world contexts while developing social, cognitive, and creative skills. Initially, students are organized into groups and subgroups, research questions are formulated by the students through brainstorming, a timeline for the completion of the activity is established, data are collected, and functional roles are assigned to the members of each group. The organization of small theatrical happenings also has significant value, as it connects environmental activities with artistic experiences. Consequently, this principle can be summarized as: **Indicative planning of activities and role-playing games, integrating environmental, mathematical, and artistic education.**

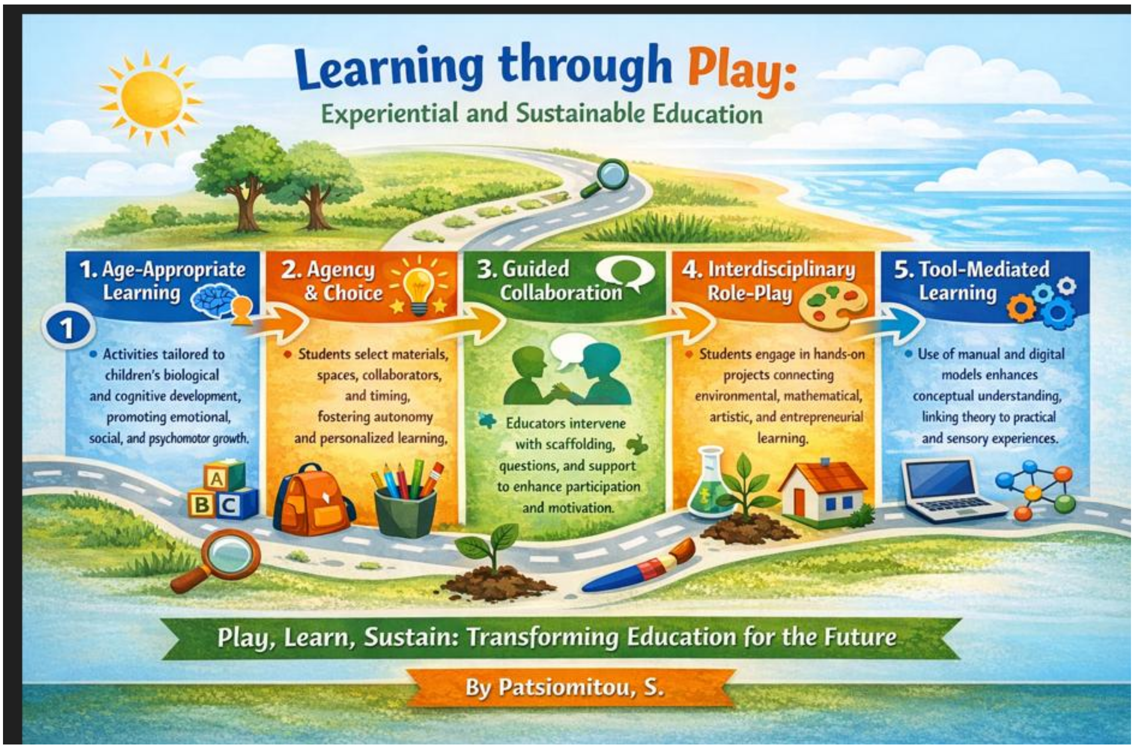

Figure 16: An overview of learning within a FDP framework generated by ChatGPT

E. The use of tools and manageable models, both manual and digital, plays a central role in supporting students' conceptual understanding. Dynamic geometry software, manipulatives, and tangible models provide multiple representations of mathematical concepts, allowing learners to explore, experiment, and reason in visual, and symbolic ways. This allows students to experience meanings, linking process and concept, theory with practice, and understanding why mathematics is meaningful in real-world contexts. By interacting with both digital and

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manual models, students integrate *procedural and conceptual knowledge* (e.g., Hiebert, & Lefevre, 1986) in a holistic and meaningful manner. Consequently, this principle can be summarized as: **Mediation through tools and manageable models, manual or digital, linking abstract concepts with embodied, visual, and experiential understanding**

Evidence from the implementation indicates that the FDP effectively supports educators in cultivating sustainability citizenship through an adaptable, interdisciplinary, and curriculum-oriented instructional design. Eventually, based on conclusions drawn from the author's previous study, the *Instructional and Learning Design for a Sustainable School* includes: (a) alignment of learning activities with children's biological maturity to support emotional, cognitive, and psychomotor development; (b) promotion of children's agency through self-selection, enabling autonomy and personalized learning pathways; (c) collaborative learning supported by timely pedagogical mediation, positioning educators as facilitators and designers of learning; (d) strategic planning of activities and role-playing scenarios integrating environmental, mathematical, and artistic education within authentic contexts; and (e) mediation through digital, dynamic, and AI-based tools that connect abstract concepts with embodied, visual, and experiential understanding.

5.0 CONCLUSIONS AND FUTURE WORK

The FDP program demonstrates that effective learning in mathematics and related disciplines can be fostered through a combination of developmental alignment, collaborative engagement, structured experiential activities, and tool-mediated exploration. By aligning tasks with students' cognitive and biological maturity, learners are able to engage meaningfully in mathematical activity without experiencing cognitive overload, thereby supporting holistic psychomotor, emotional, and linguistic development. Providing learners with autonomy and opportunities for self-selection further enhances motivation, creativity, and a sense of ownership of the learning process. Moreover, the use of both manual and digital tools enables students to explore concepts dynamically, linking theory with practice while promoting procedural, conceptual, and metacognitive understanding.

Despite the growing emphasis on interdisciplinary and sustainability-oriented education, school practices often remain fragmented, with disciplinary knowledge taught in isolation and limited opportunities for students to engage in experiential, role-based, and inquiry-driven learning. This fragmentation restricts students' ability to perceive knowledge as interconnected and to develop transferable skills across domains. Therefore, there is a need for integrative pedagogical frameworks that promote active participation, collaboration, and the meaningful synthesis of knowledge across disciplines within authentic learning contexts.

Figure 17: Utilizing ChatGPT to create a poster

At this point, I would like to make an approach for future work, related to the scenario mentioned on page 290 [with regard to the cultivation of broccoli Romanesco in school yard], focusing on the establishment of groups of children who will assume designated roles. The scenario is particularly interesting because it is interdisciplinary, and the students will be supported and guided by the respective subject teachers in order to complete it. It is important for students to approach the process of the scenario through the concept of fractal development in natural organisms in nature.

Problem Scenario (AI-Integrated Version): *Today, children, our school has decided to begin a new mission: to cultivate broccoli Romanesco in our school garden! However, we face an important problem—we do not yet know what is required for them to grow successfully. What kind of soil is most suitable? How are their growth and development influenced by water and sunlight? How quickly do they grow, and how can we systematically monitor these changes over time? Finally, how can we communicate and represent the entire process in a meaningful way so that everyone can understand it? To address this challenge, we will need a team of young experts who will investigate the problem from different perspectives. Some students will take on the role of soil scientists, exploring the characteristics of the soil and the conditions required for healthy plant growth. Others will become observers who regularly measure and record plant development. A group of students will transform these measurements into tables and diagrams in order to visualize growth patterns. Another group will act as researchers, exploring scientific information about plant development in different environmental contexts, both local and global. There will also be students responsible for documenting the process* *through photography, capturing visual evidence of change over time. Finally, some students will take on a creative role, transforming the entire experience into a story or short theatrical performance to be shared with the school community.*

During this learning journey, we will also be supported by an additional tool: an artificial intelligence system that will act as a learning assistant. This AI tool will help us search for relevant information, support us in analysing collected data, identify patterns in plant growth, compare observations over time, and assist in organizing and presenting our findings. In this way, we will not only learn about plant cultivation but also explore how digital technologies can enhance scientific inquiry and understanding. Once we organize ourselves and assign roles, our real mission will begin: to work as a collaborative team that investigates, observes, analyses, and understands the growth journey of broccoli from soil to full development.

During the implementation of the project, students are expected to observe that plant growth is not linear but structured. The broccoli, in particular, demonstrates clear fractal characteristics, where smaller parts resemble the whole structure. Data collected over time will be transformed into tables, graphs, and visual models in order to highlight these patterns. Furthermore, students will reflect on how mathematical structures such as fractals, including Sierpinski-like patterns, can be used to interpret natural growth processes. This interdisciplinary approach encourages deeper understanding of how mathematics and science are interconnected in natural systems.

Within this context, the use of Linking Visual Active Representations (LVAR) plays a central role in supporting the gradual development of students' mathematical thinking. In light of recent technological developments, the integration of *Artificial Intelligence (AI)* in mathematics education may offer new possibilities for extending the FDP framework and enhancing the pedagogical potential of LVAR process. Future research may explore several directions in relation to the incorporation of AI within such learning environments. First, it is important to examine how AI integration may influence the evolving dynamics of the didactics of mathematics. Second, research may investigate the potential of AI to support sensemaking processes and foster the development of mathematical reasoning. Third, further study is needed to understand the collaborative processes involved in incorporating AI tools into classroom practice. Finally, researchers may analyze the possible effects of AI-supported environments on students' mathematical reasoning and creative thinking.

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